Enzo MHD

Because Magnetic Fields are Awesome.

David Collins

Enzo Workshop. June 29, 2010

Two Methods

- Dedner (Wang+2009)
- CT (Collins+2010)

MHD looks just like HD...

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}(\vec{U})}{\partial x} = 0$$

...Almost...

- 7 waves
- Not Strictly Hyperbolic:
  $A(U) = \frac{\partial \vec{F}}{\partial \vec{U}}$ can have non-unique eigenvalues
- Nonconvex (Eigenvalue curvature changes sign along its characteristic)(and stuff)
- $\nabla \cdot \mathbf{B} = 0$

$$\nabla \cdot \mathbf{B} = 0$$

(the hard one)

- CT: $\partial_t \vec{B} = \nabla \times \vec{E}$
- 8 wave/Dedner: $\vec{U} = (\rho, \vec{v}, \vec{B}, E, \phi(\nabla \cdot \mathbf{B}))$
- Poisson Cleaning (Hodge Projection)

$$\vec{B}_{num} = \vec{B} + \vec{B}_{div} = \vec{B} + \nabla U$$

$$\nabla \cdot \vec{B}_{num} = \nabla^2 U$$

Components of MHD

- Flux Computation
- Making it 3d
- $\nabla \cdot \mathbf{B} = 0$ mechanism
- AMR
- Data Structures
- Altering B

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Finite Volume Basics

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}(\vec{U})}{\partial x} = 0$$

Finite Volume Basics

$$\int_x^{x+\Delta x} dx \int_t^{t+\Delta t} dt \left(\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}(\vec{U})}{\partial x} = 0 \right)$$

$$\frac{1}{\Delta t}(\hat{U}(t + \Delta t) - \hat{U}(t)) = -\frac{1}{\Delta x}(\hat{F}(x + \Delta x) - \hat{F}(x))$$

$$\frac{1}{\Delta t}(\hat{U}(t + \Delta t) - \hat{U}(t)) = -\frac{1}{\Delta x}(\hat{F}(x + \Delta x) - \hat{F}(x))$$

Volume average.
What we're looking for.

Time (and area) average.
Good for one PhD in
Applied Math.

$$\hat{U}_i^{n+1} = \hat{U}_i^n - \frac{\Delta t}{\Delta x}(\hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}})$$

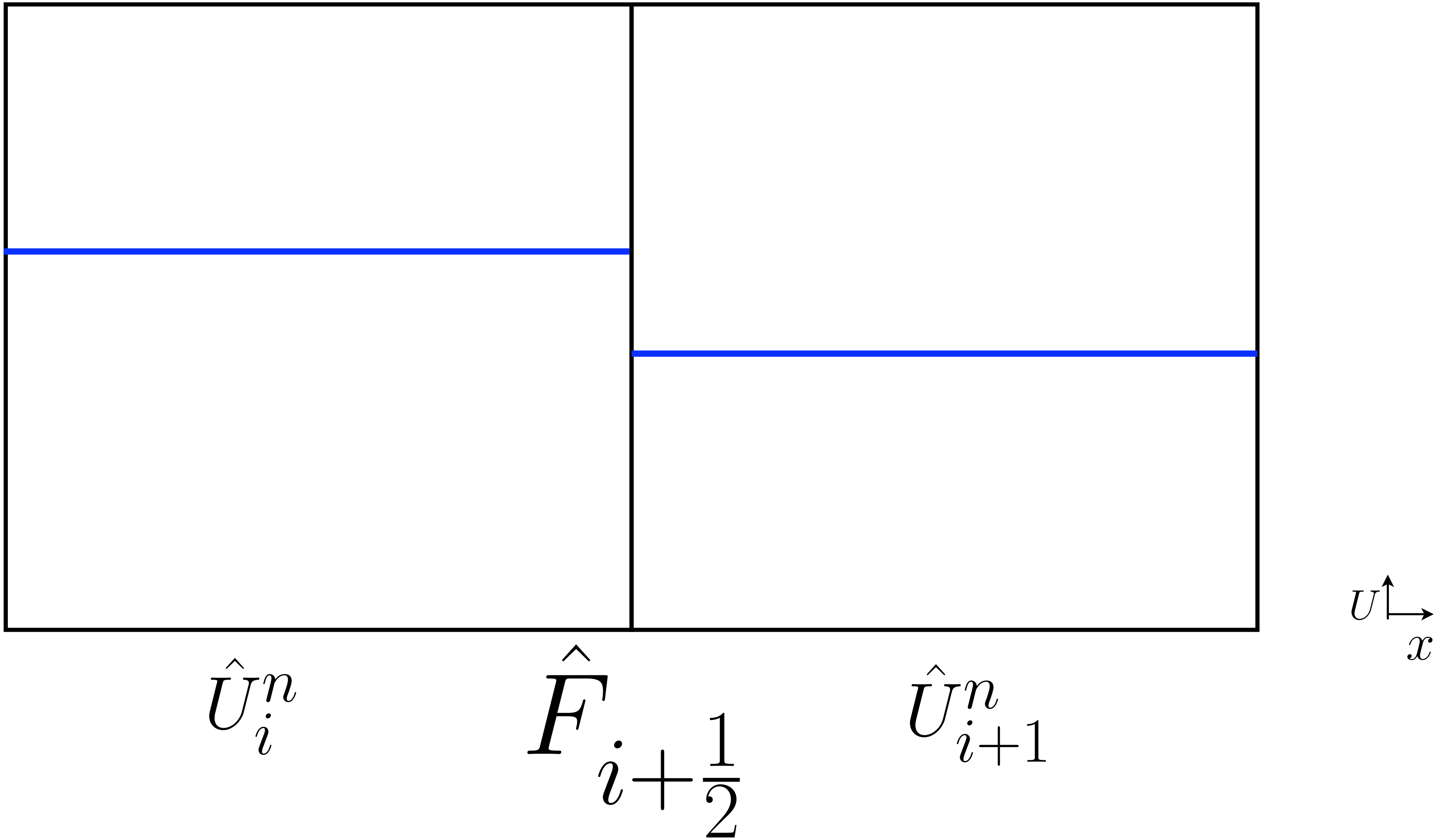
(same thing, better notation)

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \mathcal{E} \\ B_x \\ B_y \\ B_z\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho v_x \\ \rho v_x^2 + P + B^2/8\pi - B_x^2/4\pi \\ \rho v_x v_y - B_x B_y/4\pi \\ \rho v_x v_z - B_x B_z/4\pi \\ (\mathcal{E} + P + \mathbf{B}^2/8\pi)v_x - B_x(\mathbf{v}\cdot\mathbf{B})/4\pi \\ 0 \\ -E_z \\ E_y\end{pmatrix} = 0$$

$$\vec{U} \qquad\qquad \vec{F}$$

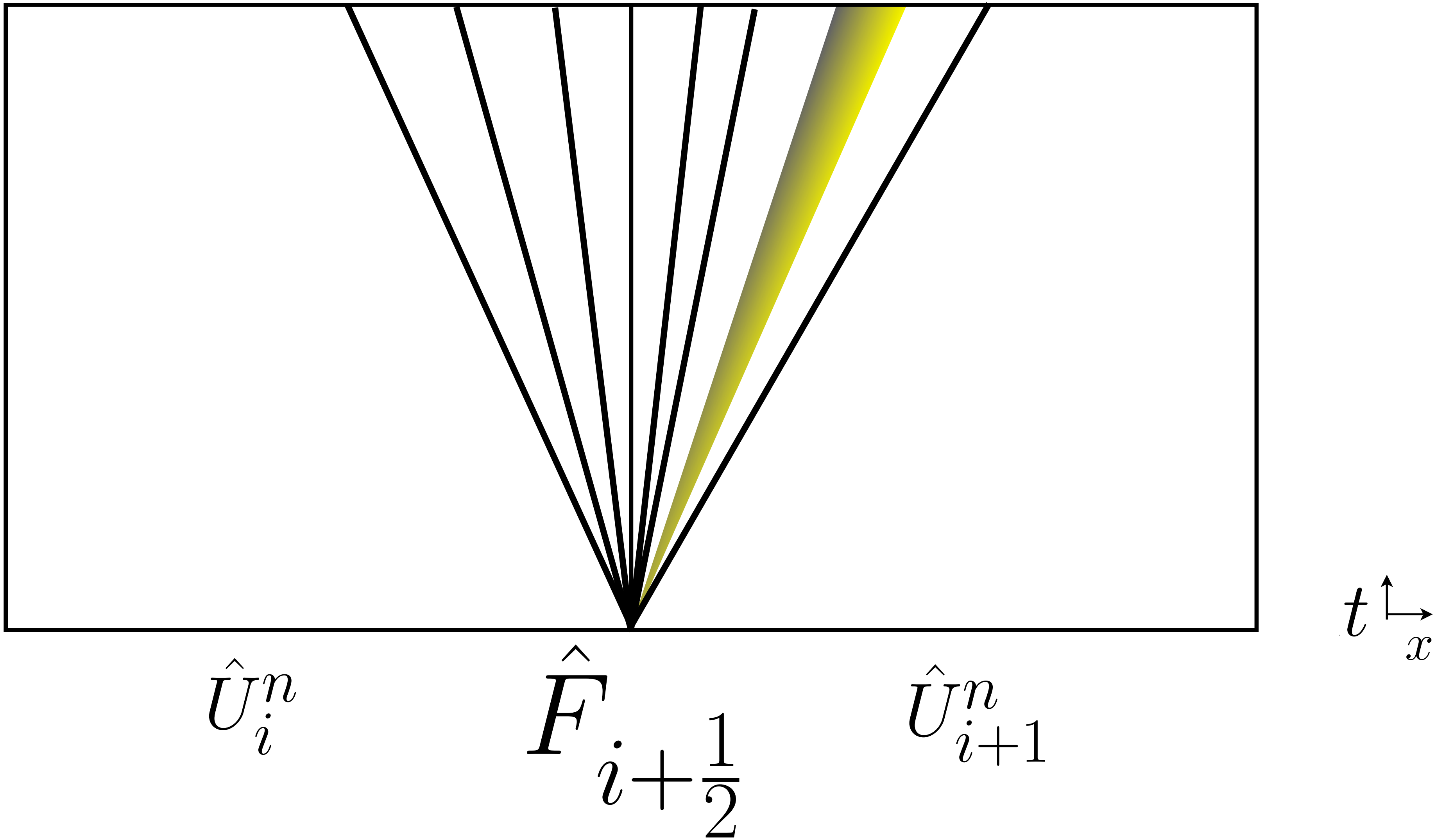
Nickel tour of Godunov

Exact Solution of
Approximate Problem



Exact Solution of
Approximate Problem

Godunov 196?



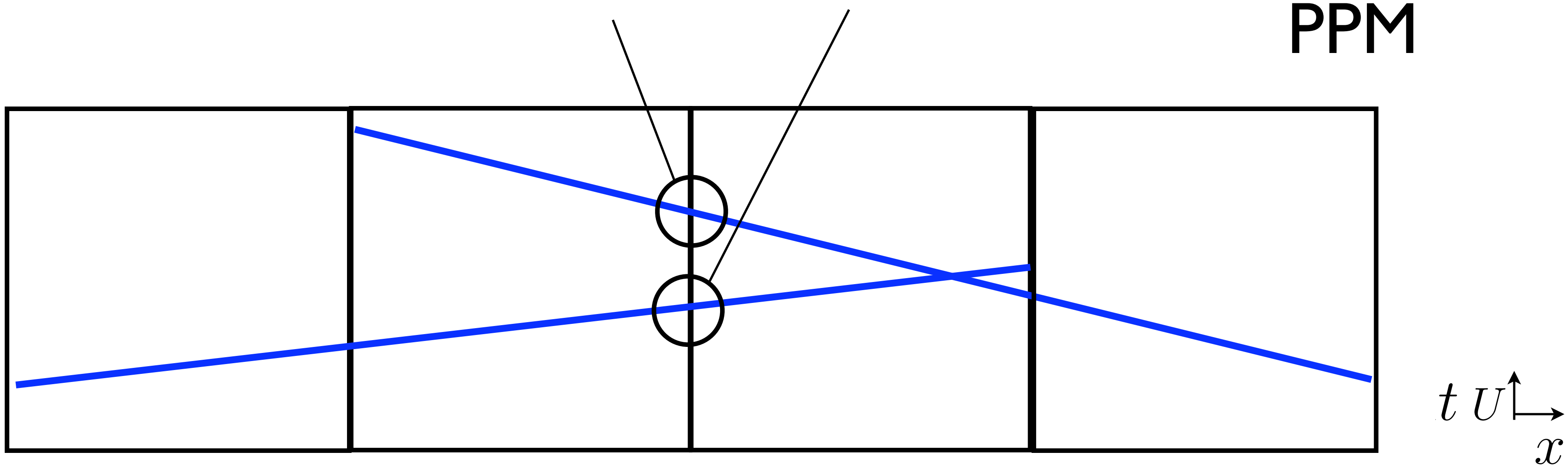
Exact Solution of
Approximate Problem

Godunov 196?

- Upwind (Reduces oscillation)
- Shock Capturing
- First Order :(

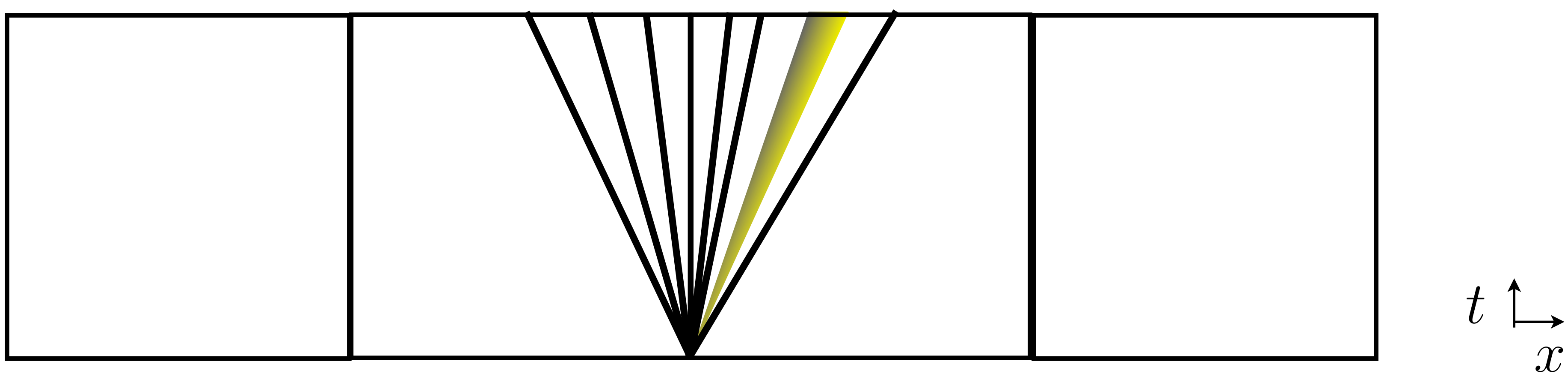
PLM, MUSCL-Hancock, PPM

$$U^{n+\frac{1}{2}}_{i+\frac{1}{2},L} \qquad U^{n+\frac{1}{2}}_{i+\frac{1}{2},R}$$

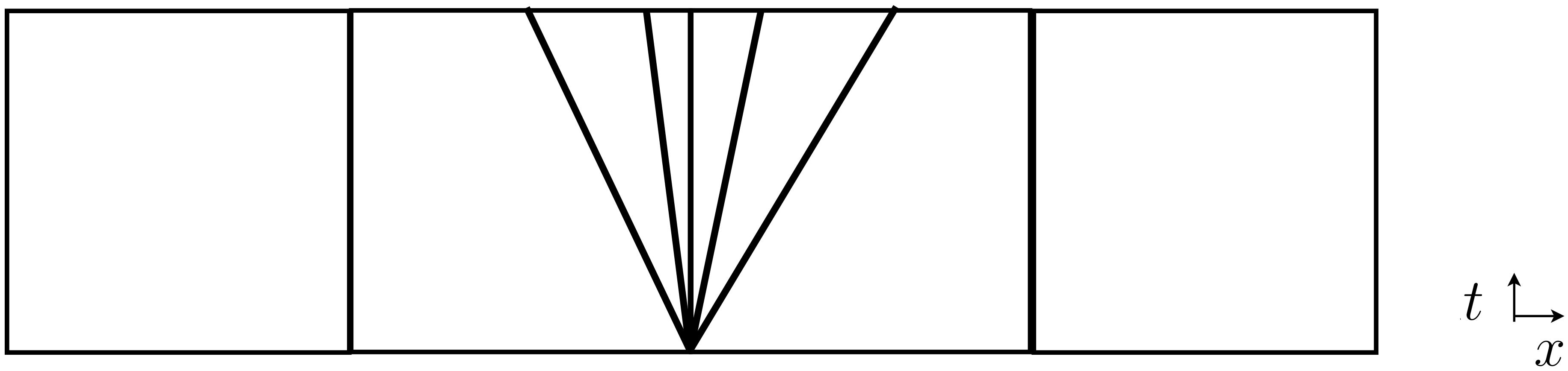


$$\hat{U}^{n+\frac{1}{2}}_{i+\frac{1}{2},L} = \hat{U}^n_i + \Delta_i \hat{U} - \partial_x F(\hat{U}_i)$$

$$\Delta_i \hat{U} = minmod(\Theta(\hat{U}_i - \hat{U}_{i-1}), \tfrac{1}{2}(\hat{U}_{i+\frac{1}{2}} - \hat{U}_{i-\frac{1}{2}}), \Theta(\hat{U}_{i+1} - \hat{U}_i))$$



But the Riemann Problem is Expensive.
Especially in MHD.

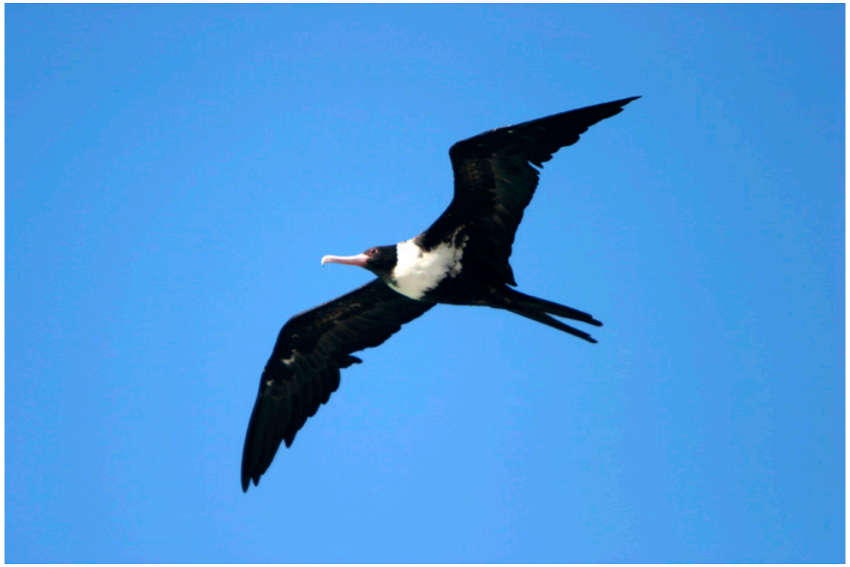


TwoShock,
HLL,
HLLD

Approximate Solution of
Approximate Problem

Convergent Evolution

- Both MHD implementations use:
  - PLM for reconstruction
  - HLL family for Riemann Solution
- Good balance between
  - Accuracy
  - Stability



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Directional Splitting

CT, PPM

$$U^* = U^n - \Delta_i F_x(U^n)$$

$$U^{**} = U^* - \Delta_j F_y(U^*)$$

$$U^{n+1} = U^{**} - \Delta_k F_z(U^{**})$$

Strang Splitting (permutation)
to reduce error

Unsplit

Dedner (MHDRK2), HydroRK2

$$
\begin{aligned}
U^{n+\frac{1}{2}} = U^n &- \Delta_i F_x(U^n) \\
&- \Delta_j F_y(U^n) \\
&- \Delta_k F_z(U^n)
\end{aligned}
$$

Unsplit

Dedner (MHDRK2), HydroRK2

Also Higher Order Time Integration!

$$
\begin{aligned}
U^{n+1} = U^{n} \quad & -\Delta_i F_x(U^{n+\frac{1}{2}}) \\
& -\Delta_j F_y(U^{n+\frac{1}{2}}) \\
& - \Delta_k F_z(U^{n+\frac{1}{2}})
\end{aligned}
$$

Requires Ghost Zone Update

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CT

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = -\nabla \cdot (\nabla \times \mathbf{E}) = 0$$

(Divergence PRESERVING.)

CT

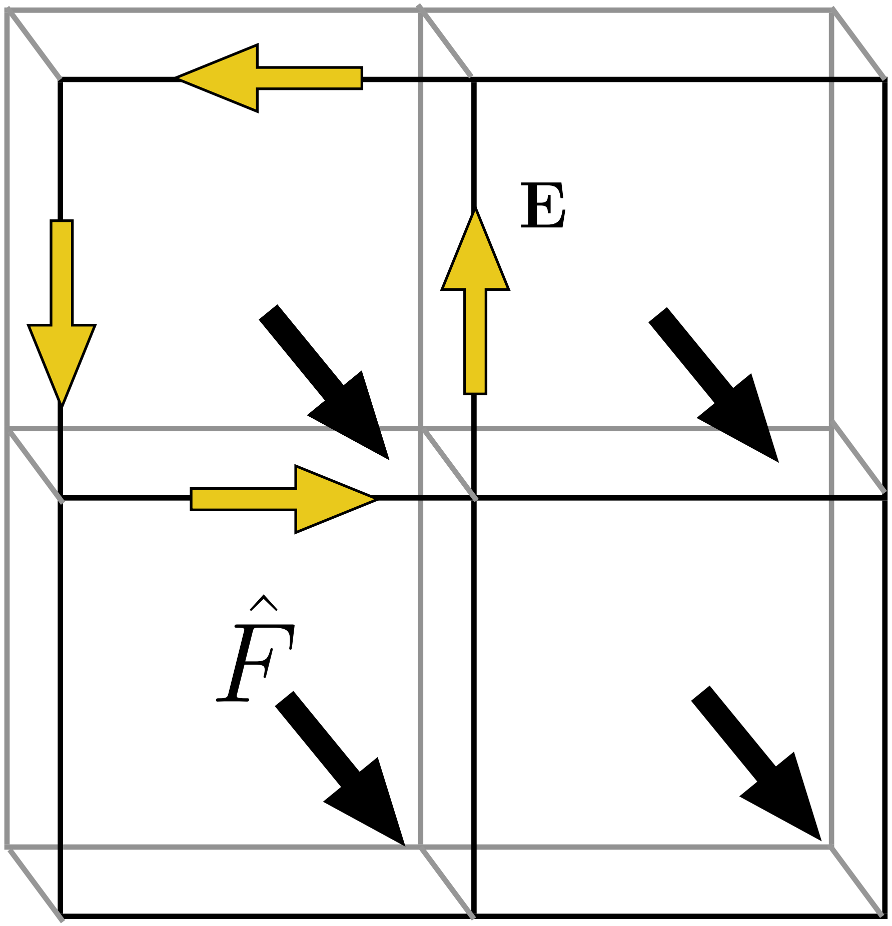
$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \mathcal{E} \\ B_x \\ B_y \\ B_z \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho v_x \\ \rho v_x^2 + P + B^2/8\pi - B_x^2/4\pi \\ \rho v_x v_y - B_x B_y/4\pi \\ \rho v_x v_z - B_x B_z/4\pi \\ (\mathcal{E} + P + \mathbf{B}^2/8\pi) v_x - B_x(\mathbf{v}\cdot\mathbf{B})/4\pi \\ 0 \\ -E_z \\ E_y \end{pmatrix}$$

$$\hat{U} \qquad \hat{F}$$

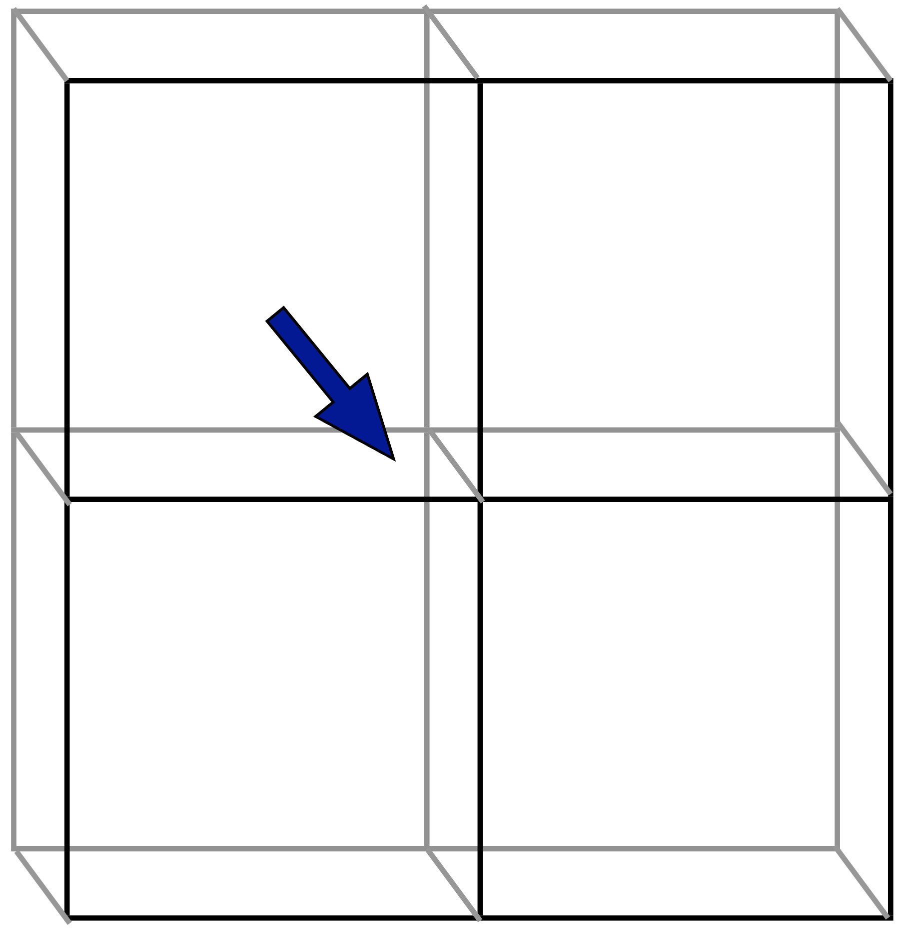
(with all the "Higher Order" goodness of Godunov)

(Balsara&Spicer 1999)

CT



Electric Field from Fluxes from Solver



$B = \nabla \times E$

(Balsara & Spicer 1999) (Gardiner & Stone 2005)

Dedner

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T) + \nabla\psi = 0$$

$$\mathcal{D}(\psi) + \nabla \cdot \mathbf{B} = 0$$

“Legrange Multiplier”

Somewhat Ad Hoc assumption of the behavior of

$$\nabla \cdot \mathrm{B}$$

$$\psi = 0 \Rightarrow \nabla \cdot \mathbf{B} = 0$$

Dedner

Leave $\nabla \cdot \mathrm{B}$
terms in Momentum & Energy eqns:

$$F_{Lorentz} = (\nabla \times B) \times B$$

(horrible vector identities from the front cover of Jackson)

$$\nabla \cdot (BB^T - \frac{1}{2}B^2\mathcal{I}) - B(\nabla \cdot B)$$

Not Removed.

(And one for energy.)

Dedner

Select

$$\mathcal{D}(\psi) := \frac{1}{c_h^2}\partial_t \psi + \frac{1}{c_p^2}\psi.$$

Then

$$\partial_t \psi + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_p^2}\psi.$$

Dedner

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u}^T + \left(p + \frac{1}{2}\mathbf{B}^2 \right) \mathcal{I} - \mathbf{B}\mathbf{B}^T \right] = -(\nabla \cdot \mathbf{B})\mathbf{B},$$

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T + \psi \mathcal{I}) = 0,$$

$$\partial_t e + \nabla \cdot \left[\left(e + p + \frac{1}{2}\mathbf{B}^2 \right) \mathbf{u} - \mathbf{B}(\mathbf{u} \cdot \mathbf{B}) \right] = -\mathbf{B} \cdot (\nabla \psi),$$

$$\partial_t \psi + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_p^2} \psi.$$

New Terms

Dedner

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u}^T + \left(p + \frac{1}{2}\mathbf{B}^2 \right) \mathcal{I} - \mathbf{B}\mathbf{B}^T \right] = -(\nabla \cdot \mathbf{B})\mathbf{B},$$

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T + \psi \mathcal{I}) = 0,$$

$$\partial_t e + \nabla \cdot \left[\left(e + p + \frac{1}{2}\mathbf{B}^2 \right) \mathbf{u} - \mathbf{B}(\mathbf{u} \cdot \mathbf{B}) \right] = -\mathbf{B} \cdot (\nabla \psi),$$

$$\partial_t \psi + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_p^2} \psi.$$

Hyperbolic! Use Godunov

Source Terms.
ψ Decays.

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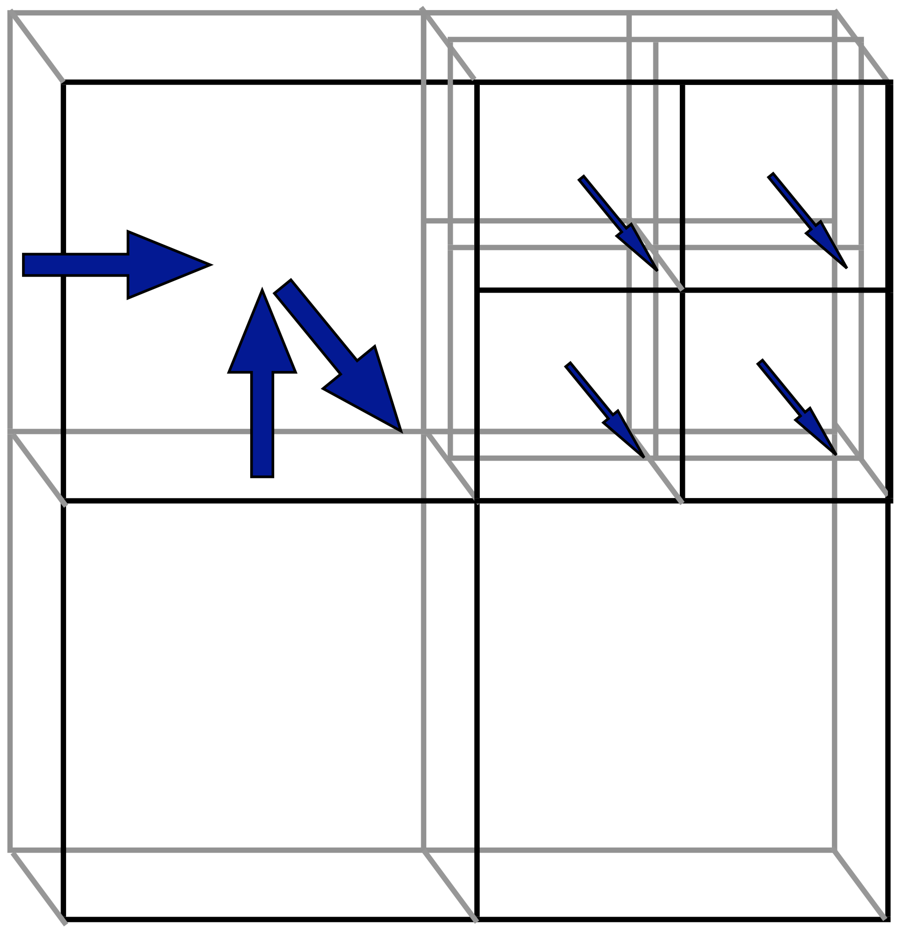
AMR: Dedner

- Use native Enzo interpolation, flux correction

AMR: CT

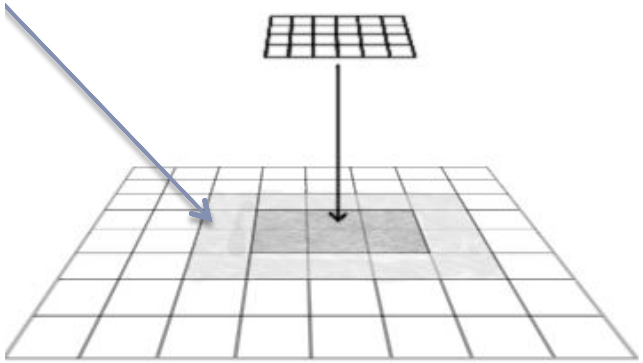


- Divergence free reconstruction

(Balsara 2001)

AMR: CT

- Flux Correction is gross and horrible, not naturally set up for all necessary cases.
- Instead, I project E, then re-curl.
- It's the identical in outcome, but easier, less error prone.



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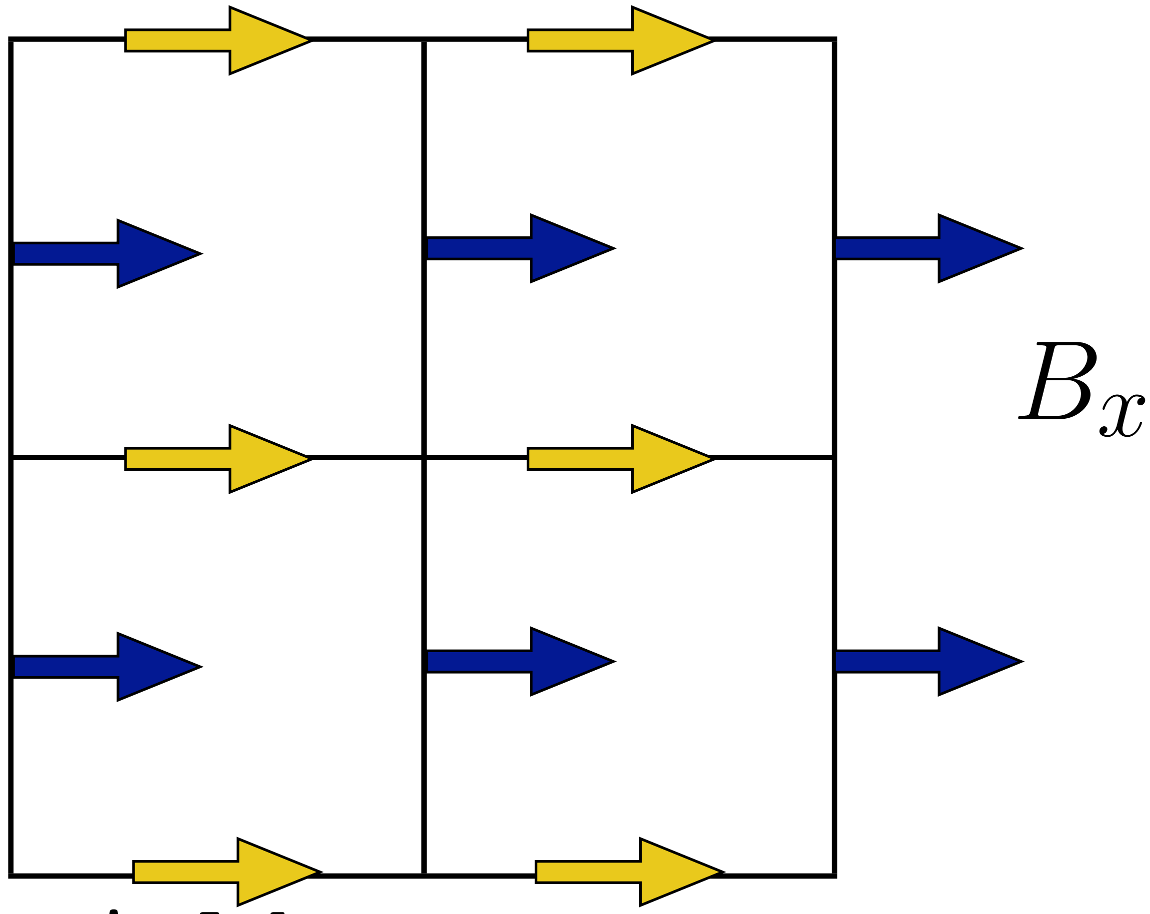
Data Structures: Dedner

- `BaryonField`
- `GradPhi`
- `EvolveLevel_RK2`

Data Structures: CT



- `MagneticField`
- `ElectricField`

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Alterations

- Dedner: Update magnetic field directly
- CT: Update electric field directly.
  (or some other divergence free addition)

Conclusions

- Talked about CT and Dedner in Enzo.
- Dedner is out now!
- Look for CT in stores soon!

Probably don't need this slide.

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u}^T + \left(p + \frac{1}{2}|\mathbf{B}|^2 \right) \mathcal{I} - \mathbf{B}\mathbf{B}^T \right] = 0$$

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T) = 0$$

$$\partial_t e + \nabla \cdot \left[\left(e + p + \frac{1}{2}|\mathbf{B}|^2 \right) \mathbf{u} - \mathbf{B}(\mathbf{u} \cdot \mathbf{B}) \right] = 0$$