Core Collapse &
Neutron Star Mergers

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Vote:

What provides the pressure that stabilizes neutron stars against gravity?

(a) Neutron degeneracy.

(b) Mixture of neutron and proton degeneracy.

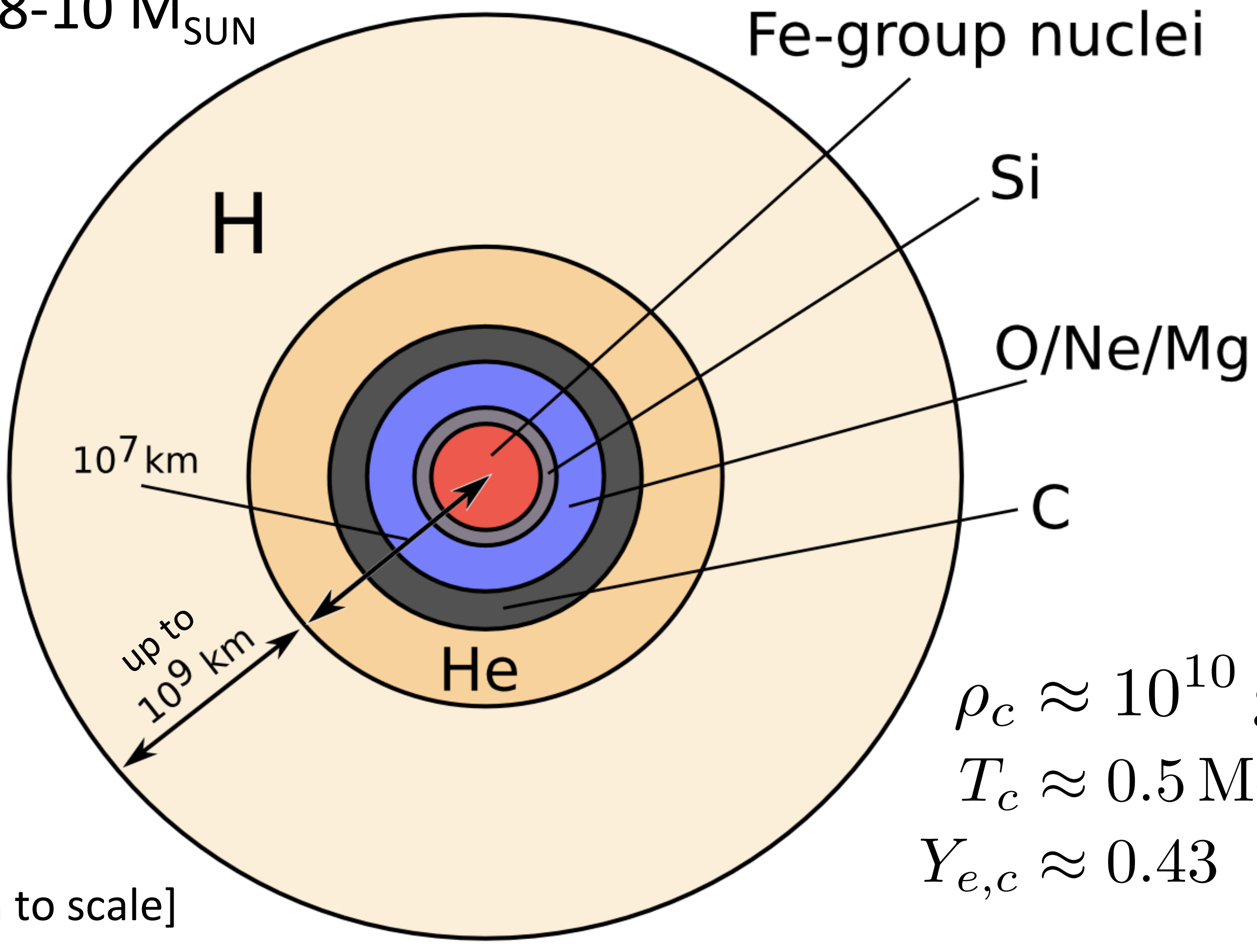
(c) None of the above.

Lecture Plan

- Lecture 1 (now!)
  - Core collapse supernovae (CCSNe), the nuclear equation of state, and neutron star structure.
  - Numerical relativity, general-relativistic hydrodynamics, and neutron star merger simulations with the `Einstein Toolkit`.
- "Workshop" (this afternoon)
  - Neutron star structure calculations
  - Black hole formation in stellar collapse
  - Neutron star merger simulations
- Lecture 2 (tomorrow!)
  - LIGO and Gravitational-Wave Astronomy
  - Phenomenology of neutron star mergers.
  - Extreme core collapse events and the CCSN-LGRB relationship.
  - Gravitational waves from core-collapse supernovae.

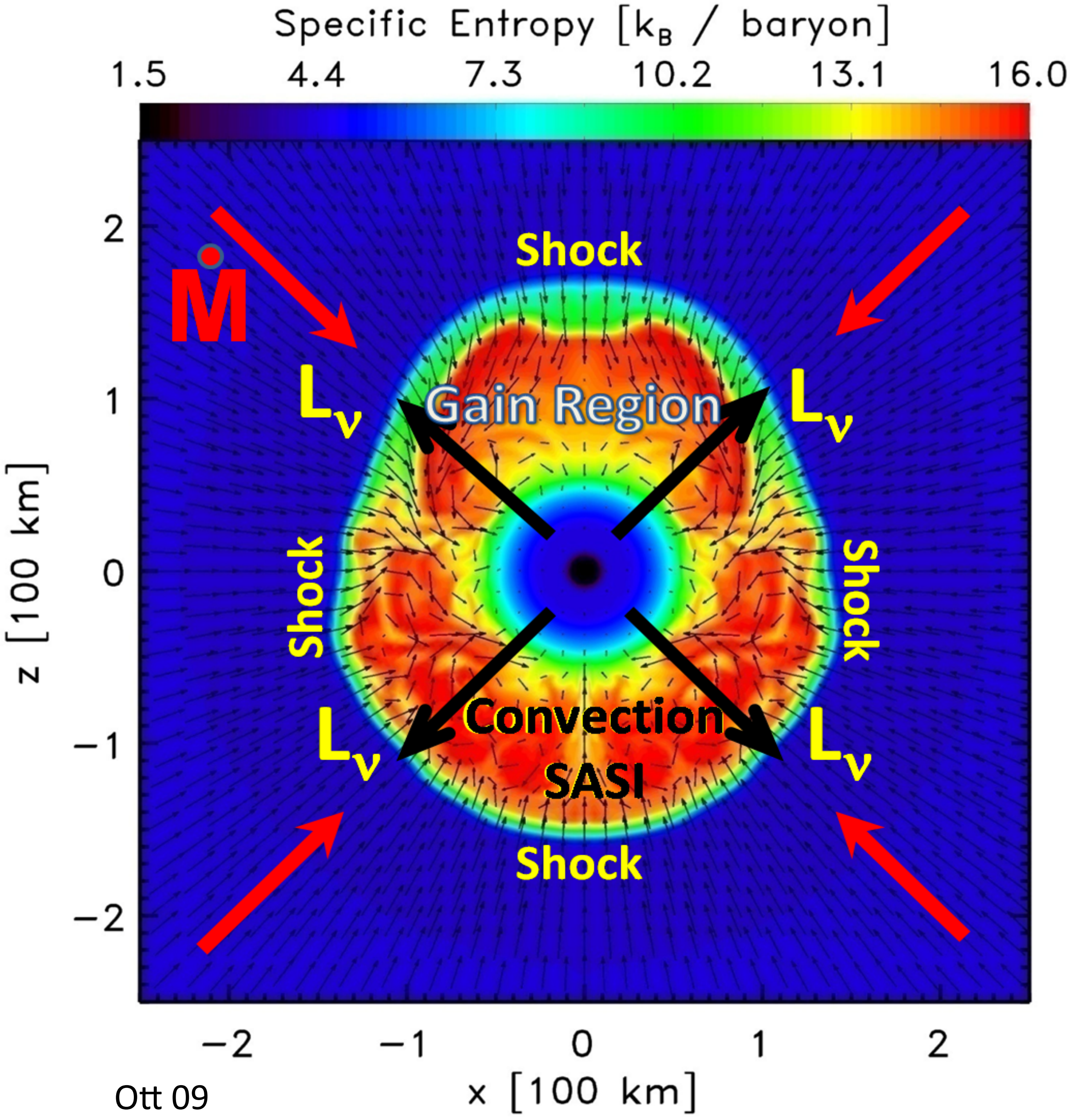
Core Collapse

$M_{initial}$ > 8-10 M_{SUN}



$$\rho_c \approx 10^{10}\,\mathrm{g\,cm^{-3}}$$
$$T_c \approx 0.5\,\mathrm{MeV}$$
$$Y_{e,c} \approx 0.43$$

[not drawn to scale]



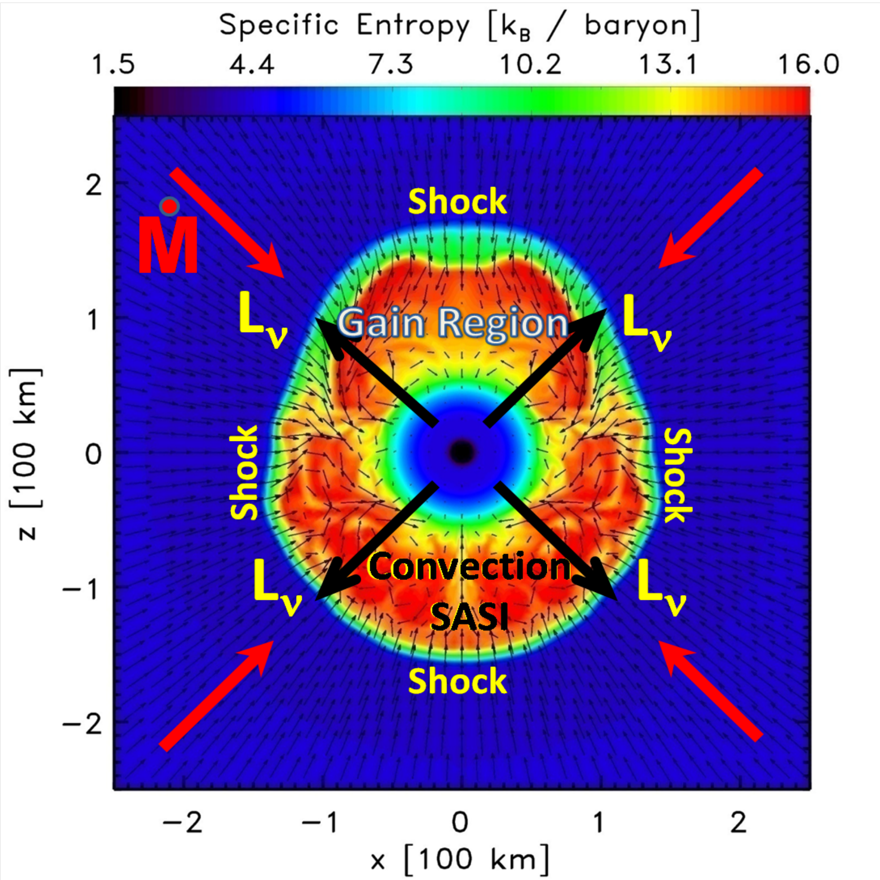
Ott 09

CCSN

160 ms after core bounce.

2D simulation of Ott+08

What are the Physics Ingredients?

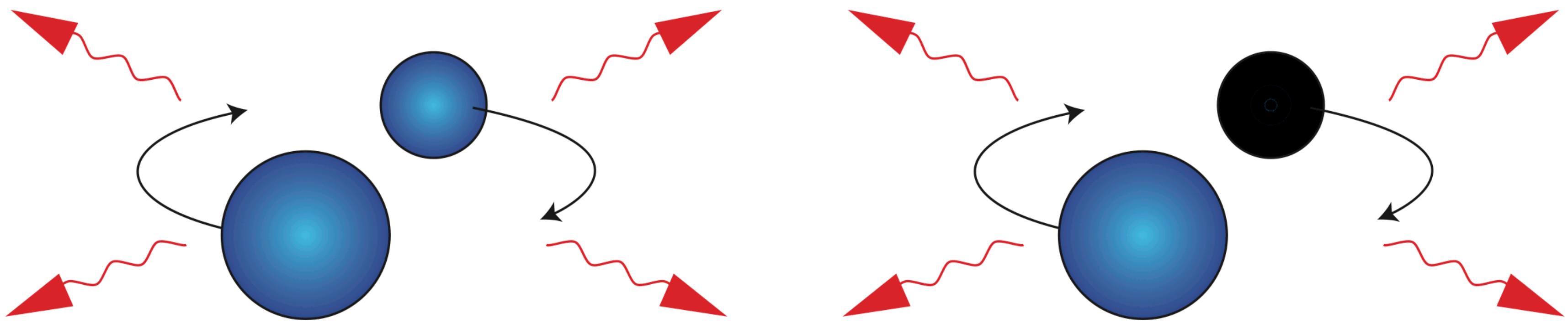


- Gravity
- Nuclear physics / nuclear equation of state / nuclear reactions (strong force)
- Neutrino physics (weak force)
- Fluid dynamics / MHD (EM)
- Transport theory

Neutron Star Mergers

- Neutron Star + Neutron Star (NSNS)
- Black Hole + Neutron Star (BHNS)

credit: J. Read

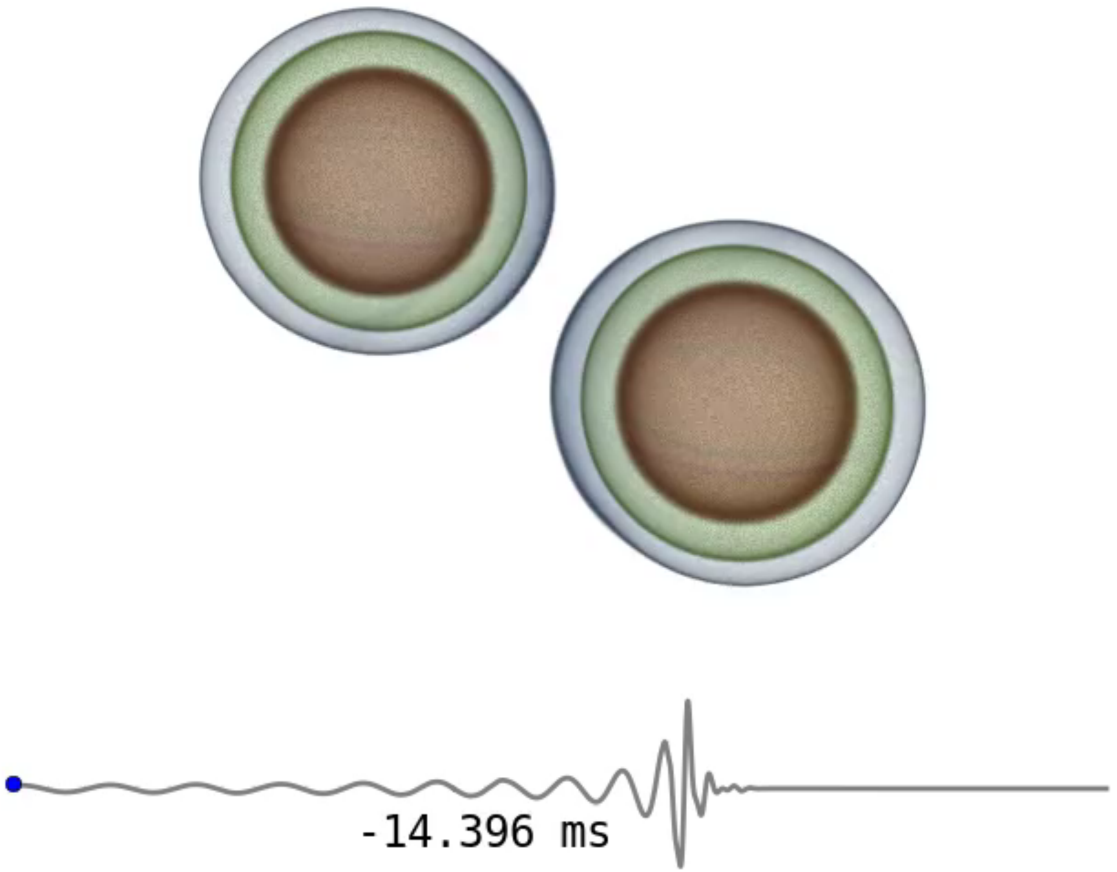


$M_1 \sim M_2 \sim 1.4\ M_{Sun}$
-> galactic NSNS binaries!

$M_{BH} \sim 7\text{-}10 \times M_{NS}$ (Belczynski+10)
(but no BHNS systems known)

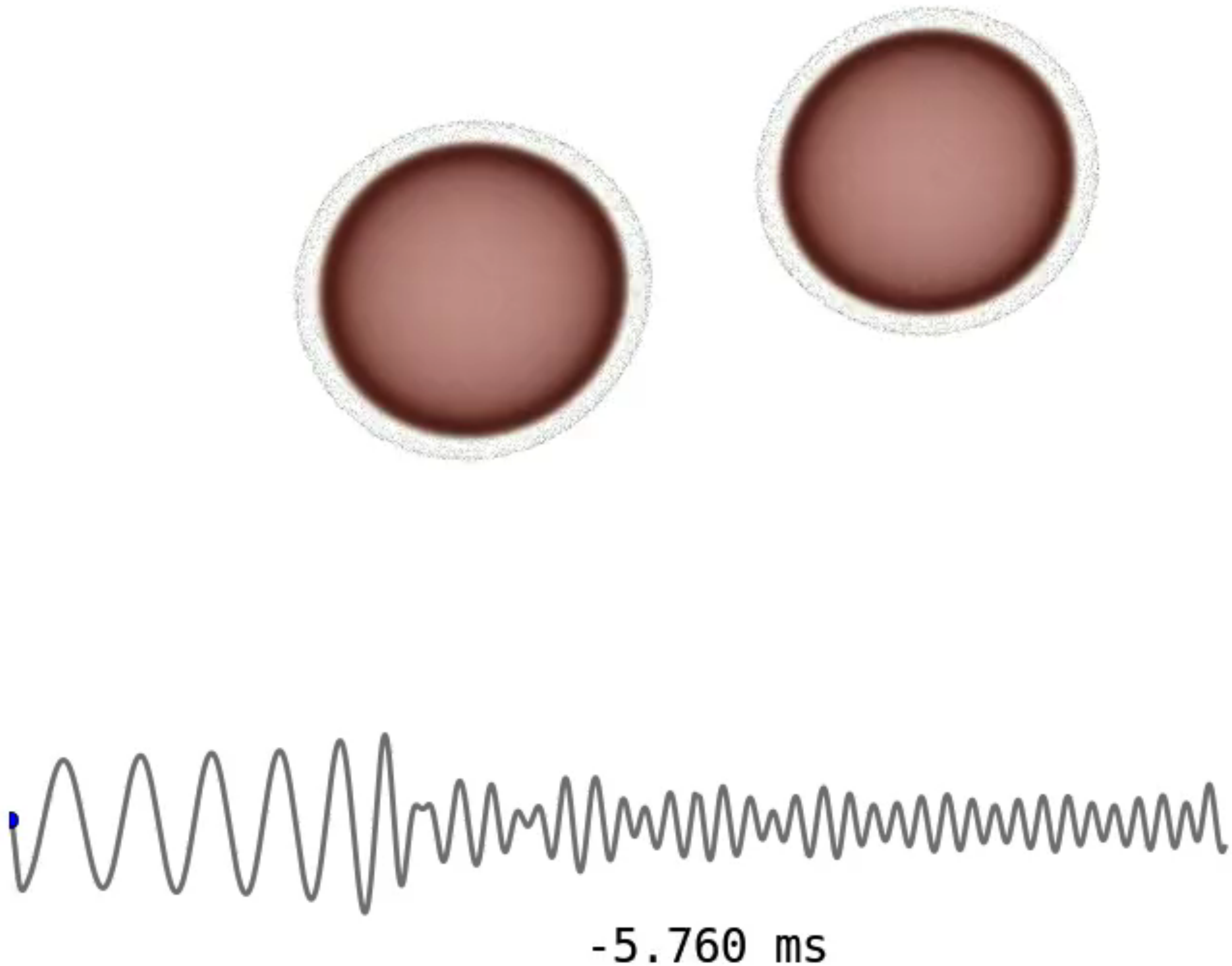
- Inspiral driven by gravitational-wave (GW) emission.

Double Neutron Star Mergers: Case A



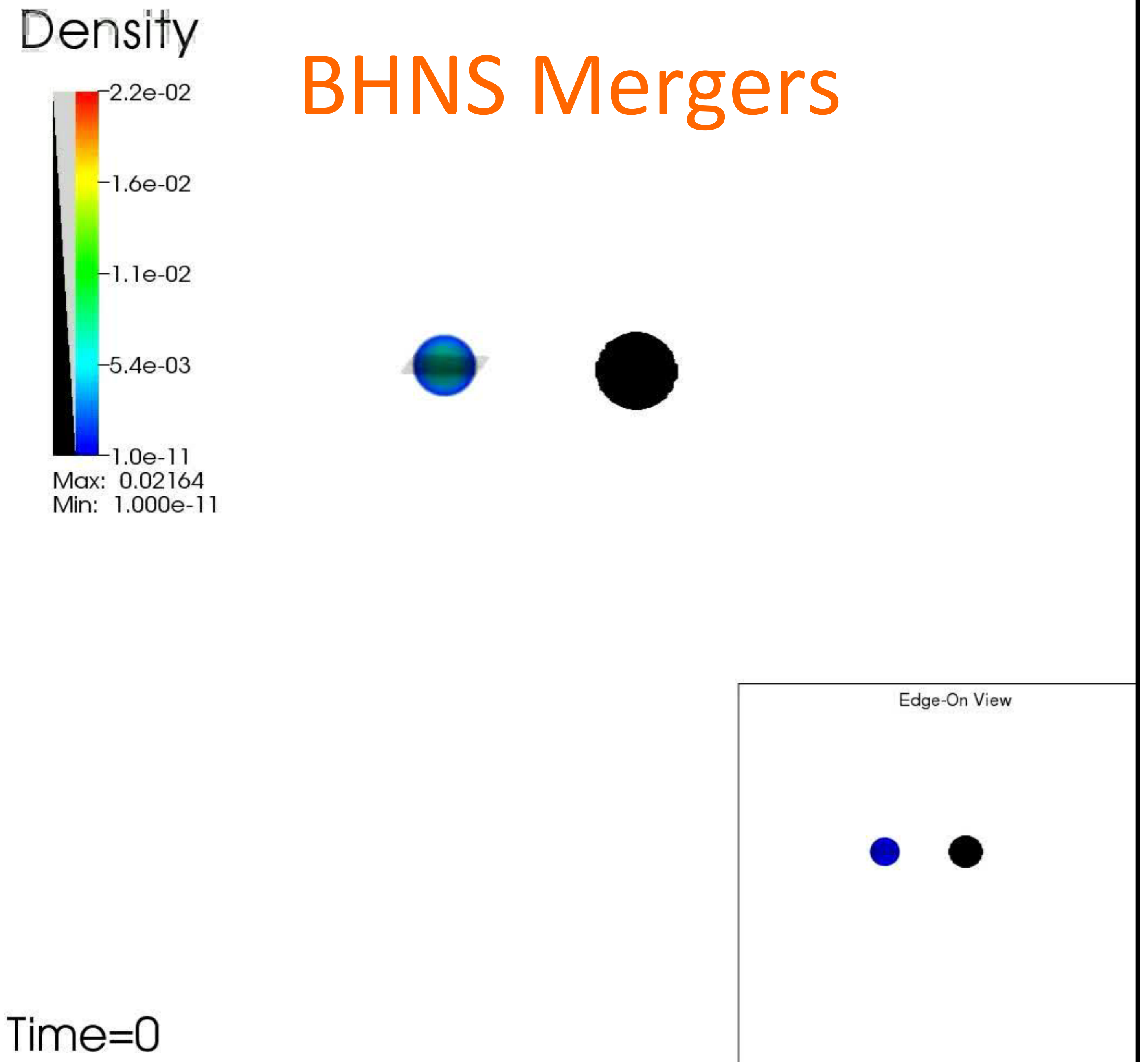
credit:
R. Haas, SXS

Double Neutron Star Mergers: Case B



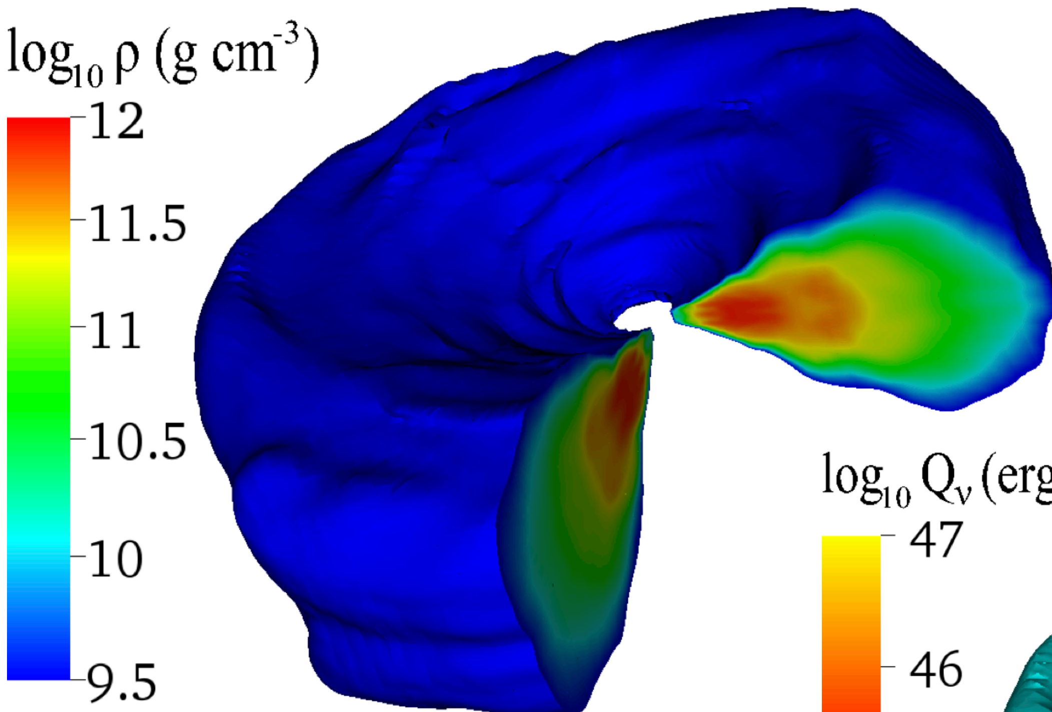
credit:
R. Haas, SXS

BHNS Mergers



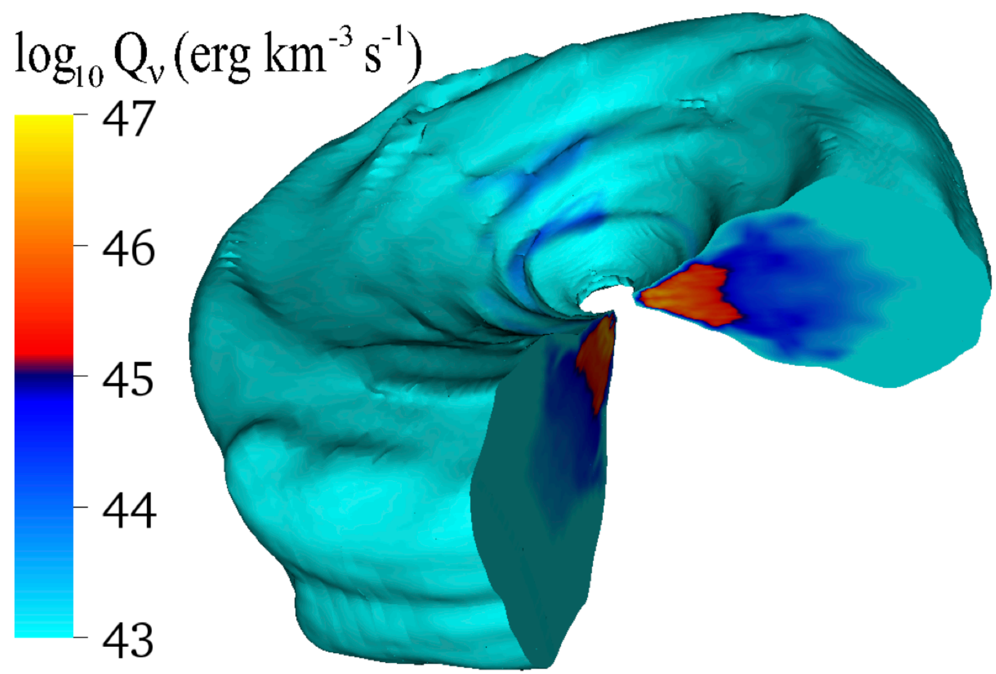
Duez+
SXS

Postmerger Disks



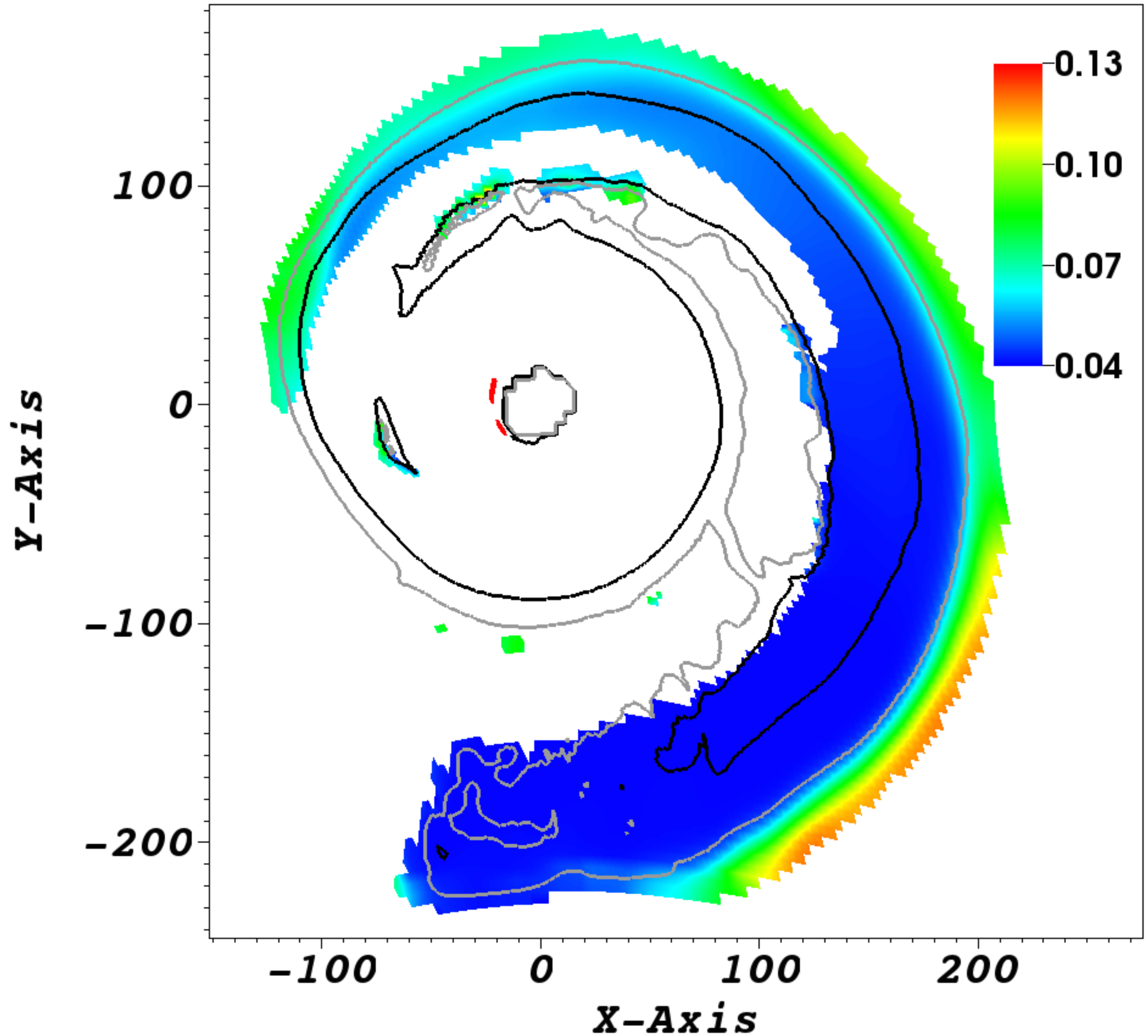
$$\tau \lesssim 15$$

$$T_{\mathrm{cool}} \sim \frac{E_{\mathrm{thermal}}}{L_\nu} \approx 10\,\mathrm{ms}\text{–}100\,\mathrm{ms}$$



30 ms after merger
Radius ~110 km
Thickness ~45 km

r-Process Nucleosynthesis

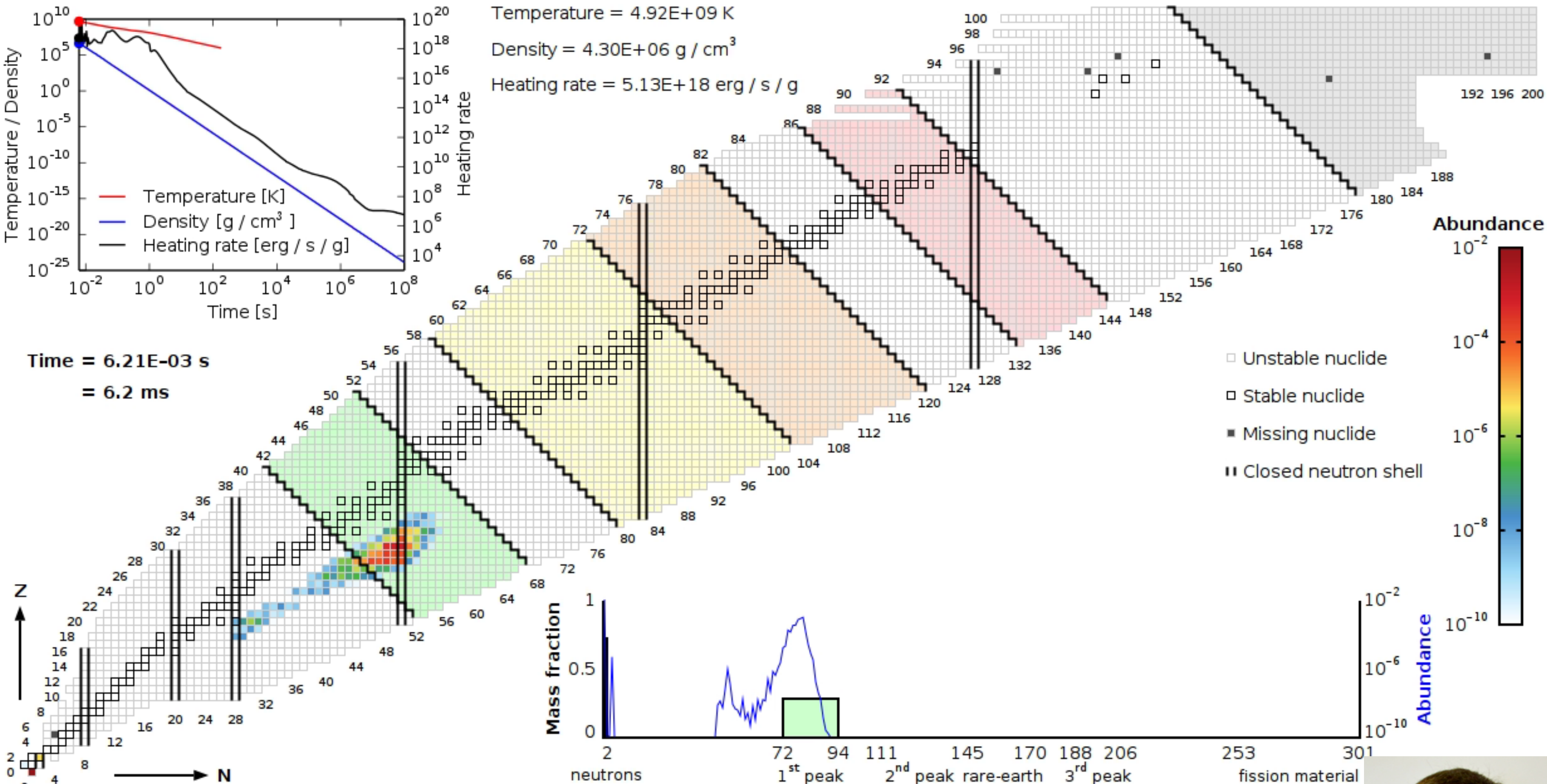


BHNS merger
Foucart+14

Electron fraction Y_e

credit:
J. Lippuner, SXS

r-Process Nucleosynthesis



Merger outflows: very neutron rich material

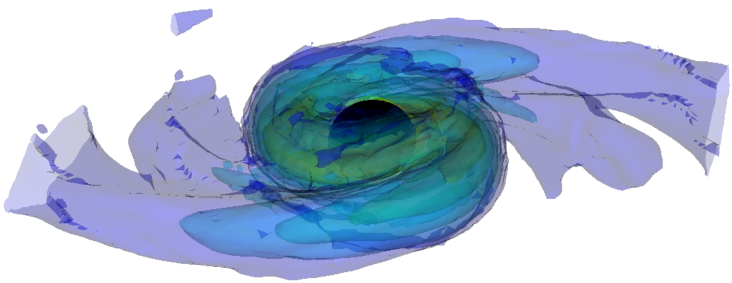
See lectures by Qian, Kasen, Fröhlich

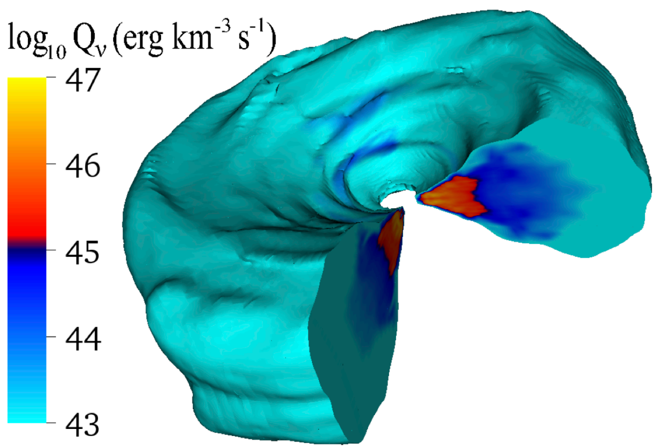
Caltech

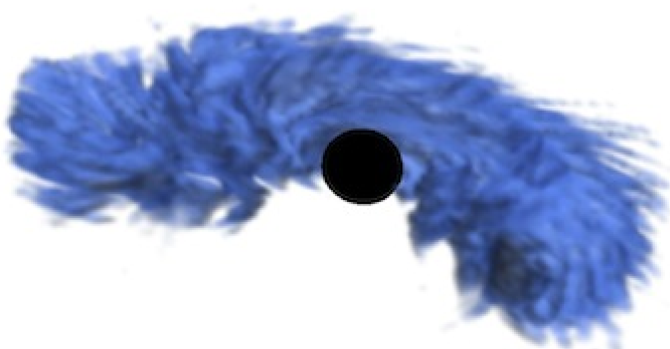
credit:

J. Lippuner, SXS

What are the Physics Ingredients?







- Gravity
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- Neutrino physics (weak force)
- Fluid dynamics / MHD (EM)
- Transport theory

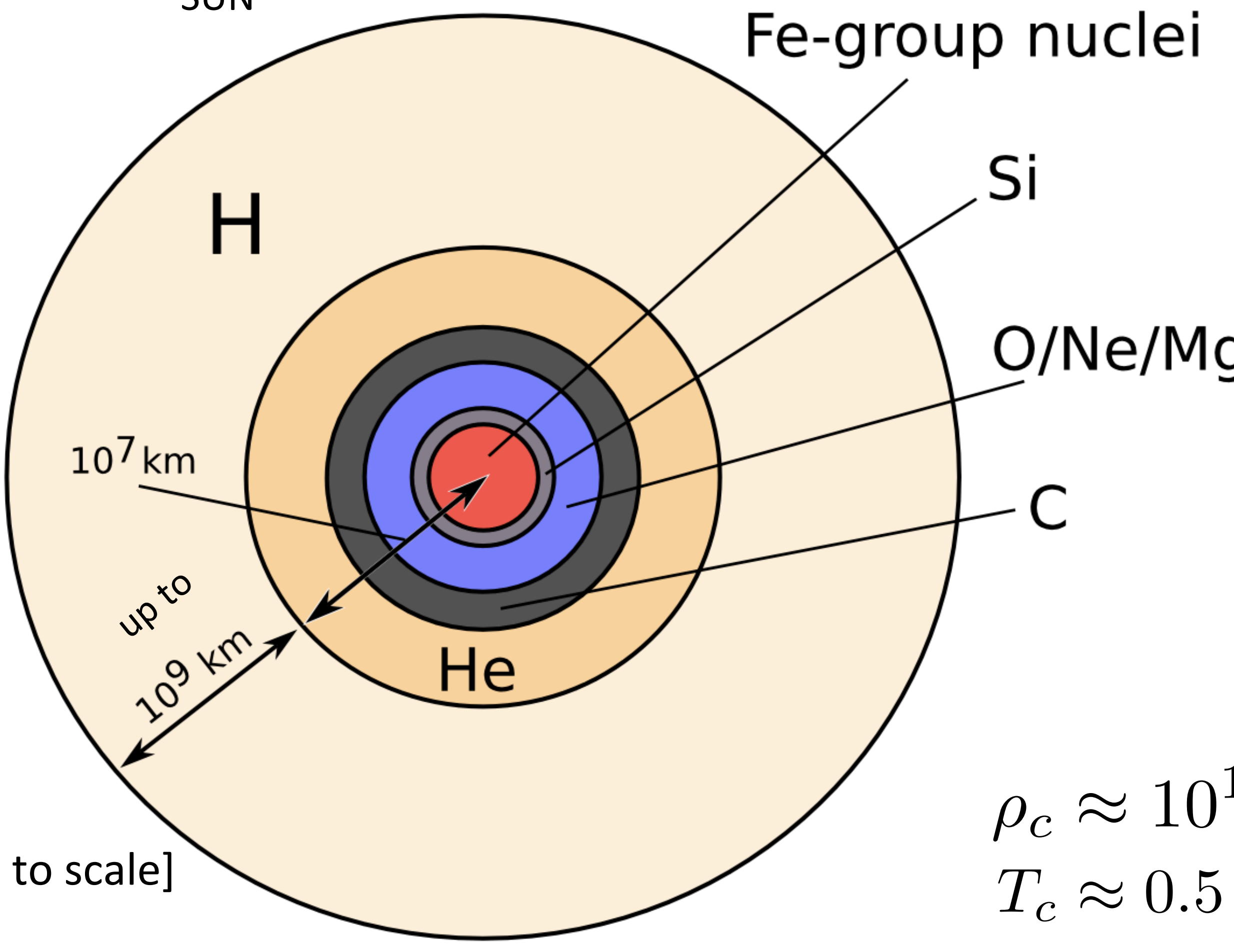
Take Away:

- Core-collapse supernovae and neutron star mergers involve the same rich physics.
- Both are cosmic laboratories for fundamental physics.
- Both are 3D multi-scale problems.

Core Collapse, The Nuclear Equation of State, and Neutron Star Structure

Core Collapse

$M_{\text{initial}} > 8\text{-}10\ M_{\text{SUN}}$



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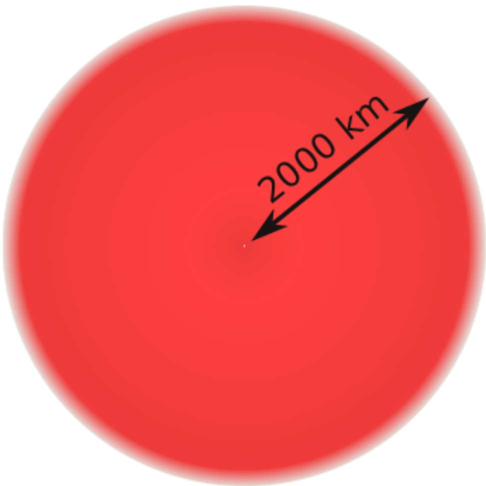
Hydrostatics of the Iron Core



Iron Core

$\rho_c \approx 10^{10}$ g/cm^3

$T \approx 1$ MeV

$Y_e \approx 0.5$

(in reality: T lower and Y_e slightly lower)

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

What produces the pressure?

ions (iron-group nuclei)

electrons

photons

$$P = P_{\rm ion} + P_{\rm rad} + P_e$$

What dominates?

Ion EOS in the Iron Core

- Ideal Boltzmann gas of non-interacting particles.

$$P_{\mathrm{ion}} = n_{\mathrm{ion}} kT \quad n = \frac{\rho}{\mu m_u} \quad \mu = \left(\sum_i \frac{X_i}{A_i} \right)^{-1}$$

For pure, say, ^{56}Ni: $\mu = 56$

$$P_{\mathrm{ion}} = \frac{\rho N_A}{56} kT = 1.7 \times 10^{26} \left(\frac{\rho}{10^{10}\,\mathrm{g\,cm^{-3}}} \right) \left(\frac{T}{1\,\mathrm{MeV}} \right) \mathrm{dyn\,cm^{-2}}$$

Photon EOS in the Iron Core

- Ideal Bose gas:

$$P_\gamma = \frac{1}{3} a T^4 = 4.6 \times 10^{25} \left(\frac{T}{1\,\mathrm{MeV}} \right)^4 \,\mathrm{dyn\,cm}^{-2}$$

Electron EOS in the Iron Core

- Ideal Fermi gas, but electrons are *relativistic* and *degenerate*:

$$\eta = \frac{\mu_e}{kT} \gg 1 \qquad \beta = \frac{kT}{m_e c^2} \gg 1$$

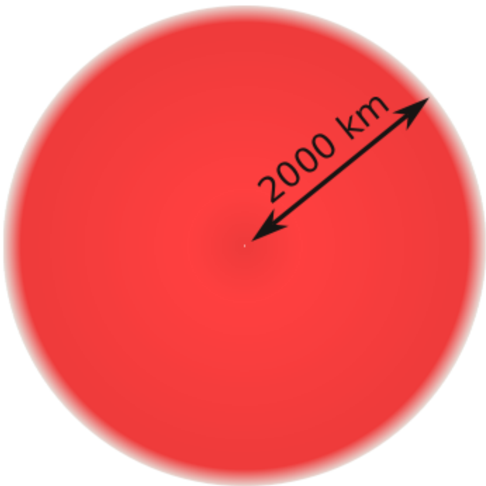
degeneracy parameter relativity parameter

In this case:

$$P_e = K\rho^\gamma = 1.2435 \times 10^{15} Y_e^{4/3} \rho^{4/3}$$

$$P_e = 10^{28} \left(\frac{Y_e}{0.5}\right)^{4/3} \left(\frac{\rho}{10^{10}\,\mathrm{g\,cm^{-3}}}\right)^{4/3} \mathrm{dyn\,cm^{-2}}$$

Equation of state in the Iron Core



Iron Core

$\rho_c \approx 10^{10}$ g/cm^3
T ≈ 1 MeV
$Y_e \approx 0.5$

(in reality: T lower and Y_e slightly lower)

$$P = P_{\rm ion} + P_{\rm rad} + P_e$$

$$P_{\rm ion} = 1.7 \times 10^{26} \left(\frac{\rho}{10^{10}\,{\rm g\,cm^{-3}}}\right)\left(\frac{T}{1\,{\rm MeV}}\right)\ {\rm dyn\,cm^{-2}}$$

$$P_\gamma = \frac{1}{3} a T^4 = 4.6 \times 10^{25} \left(\frac{T}{1\,{\rm MeV}}\right)^4\ {\rm dyn\,cm^{-2}}$$

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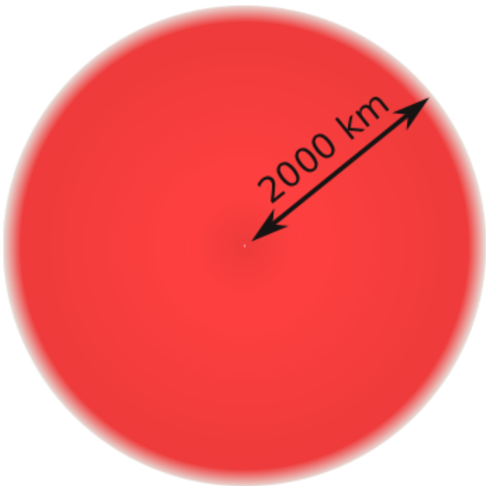
$$P_e \gg P_{\rm ion} \gg P_{\rm rad}$$

Onset of Collapse



Iron Core

$\rho_c \approx 10^{10}$ g/cm^3

$T \approx 1$ MeV

$Y_e \approx 0.5$

(in reality: T lower and Y_e slightly lower)

- Chandrasekhar:

$$M_{\mathrm{Ch,eff}} \approx 1.44 \left(\frac{Y_e}{0.5}\right)^2 M_\odot$$

+ corrections: GR, thermal, surface P etc.

No equilibrium solutions exists for relativistic & degenerate electron gas for

$$M > M_{\mathrm{Ch,eff}}$$

-> radial instabilty -> core collapse!

Two ways to get there:

(1) Silicon shell burning adding mass to the core.

(2) Reduction of Y_e.

-> electron capture

Nuclear Statistical Equilibrium

- At high temperature (> 0.5 MeV), strong forward and backward reactions between nuclei and nucleons proceed rapidly.
- "Chemical equilibrium" is reached:

$$Z_i \mu_p + N_i \mu_n = \mu_i$$

$$n = \sum_i n_i A_i$$

Mass conservation:
total number density of nulceons

$$n Y_e = n_p + 2 n_\alpha + \sum_i Z_i n_i$$

Charge conservation

- Leads to a set of Saha-like equations for abundances $Y_i = \dfrac{n_i}{n}$

nuclear part. fn.

$$Y_{Z_i,A_i} = \frac{G_{Z_i,A_i}}{2^A (m_u kT/(2\pi\hbar)^{(3/2[A-1])}} (\rho N_A)^{A-1} Y_p^Z Y_n^N \exp\left(\frac{Q}{kT}\right)$$

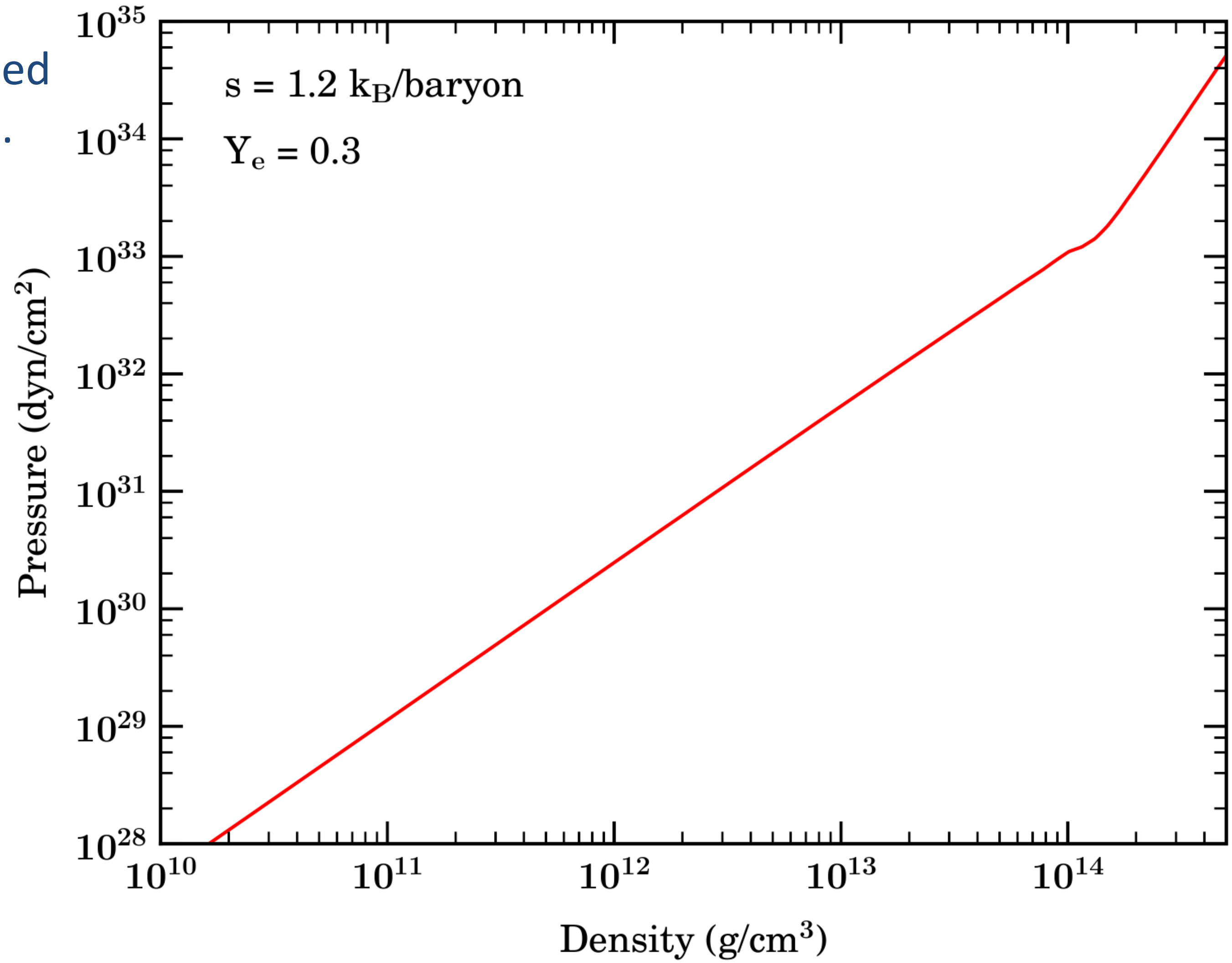
$$Q = Z m_p + N m_n - M(N,Z) \qquad A = N + Z$$

Equation of State in Collapse

Nuclear Statistical Equilibrium ($\rho > 10^7$ g/cm^3, T > 0.5 MeV)

-> $P = P(\rho, T, Y_e)$

Composition determined by Saha-type equation.



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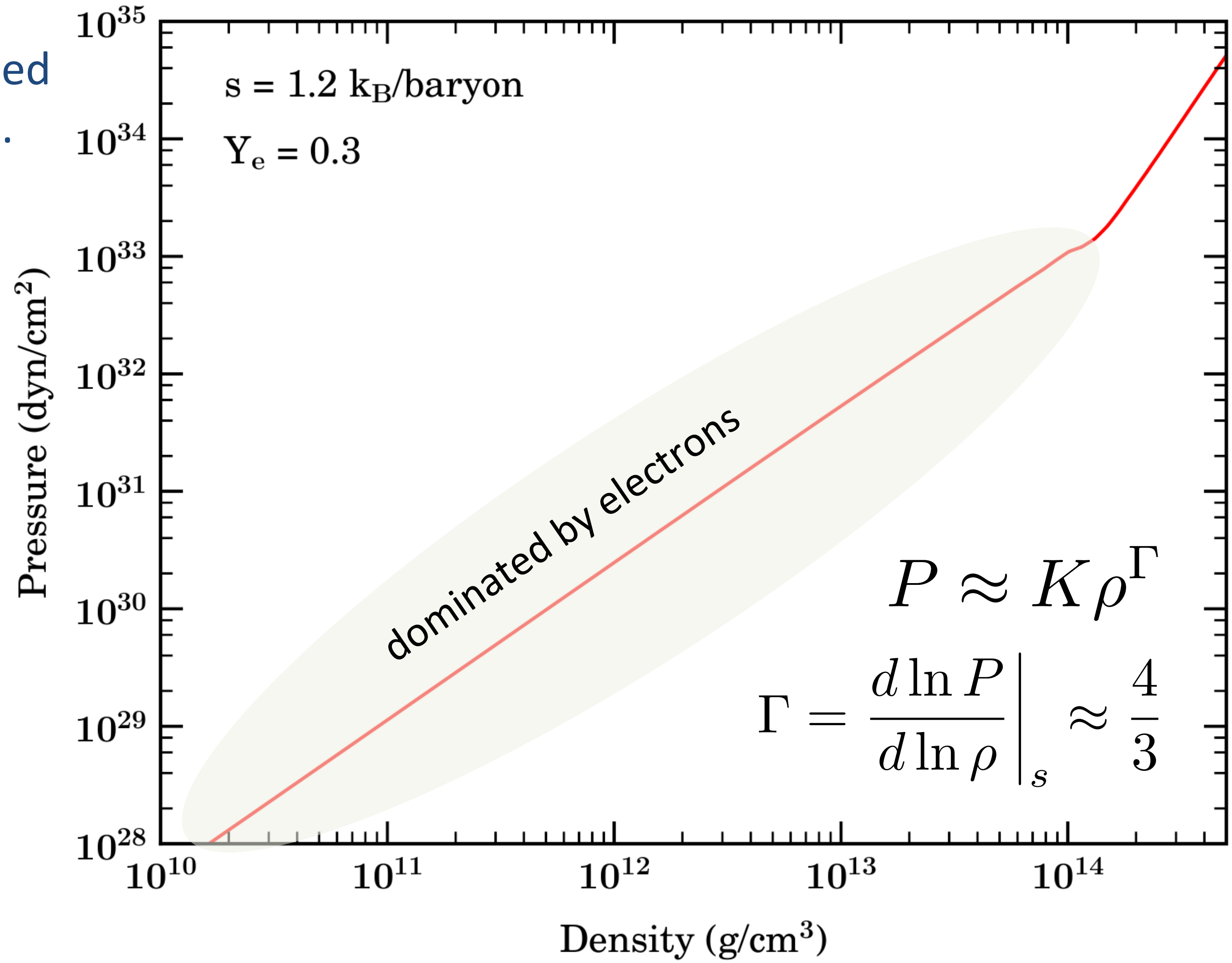

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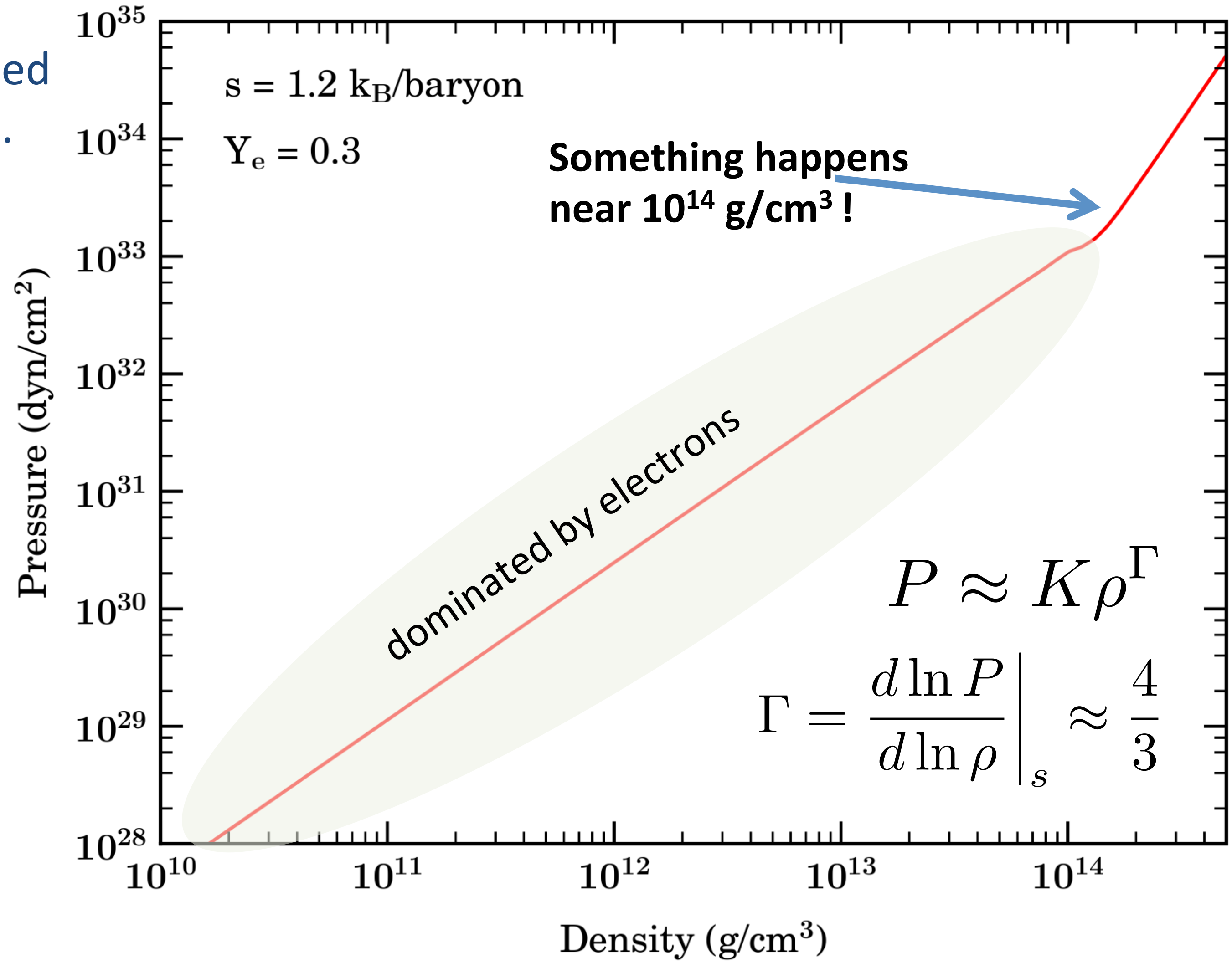


Nuclear Equation of State

Nuclear Physics:

$$R_{\rm nuc} = A^{1/3} r_0$$

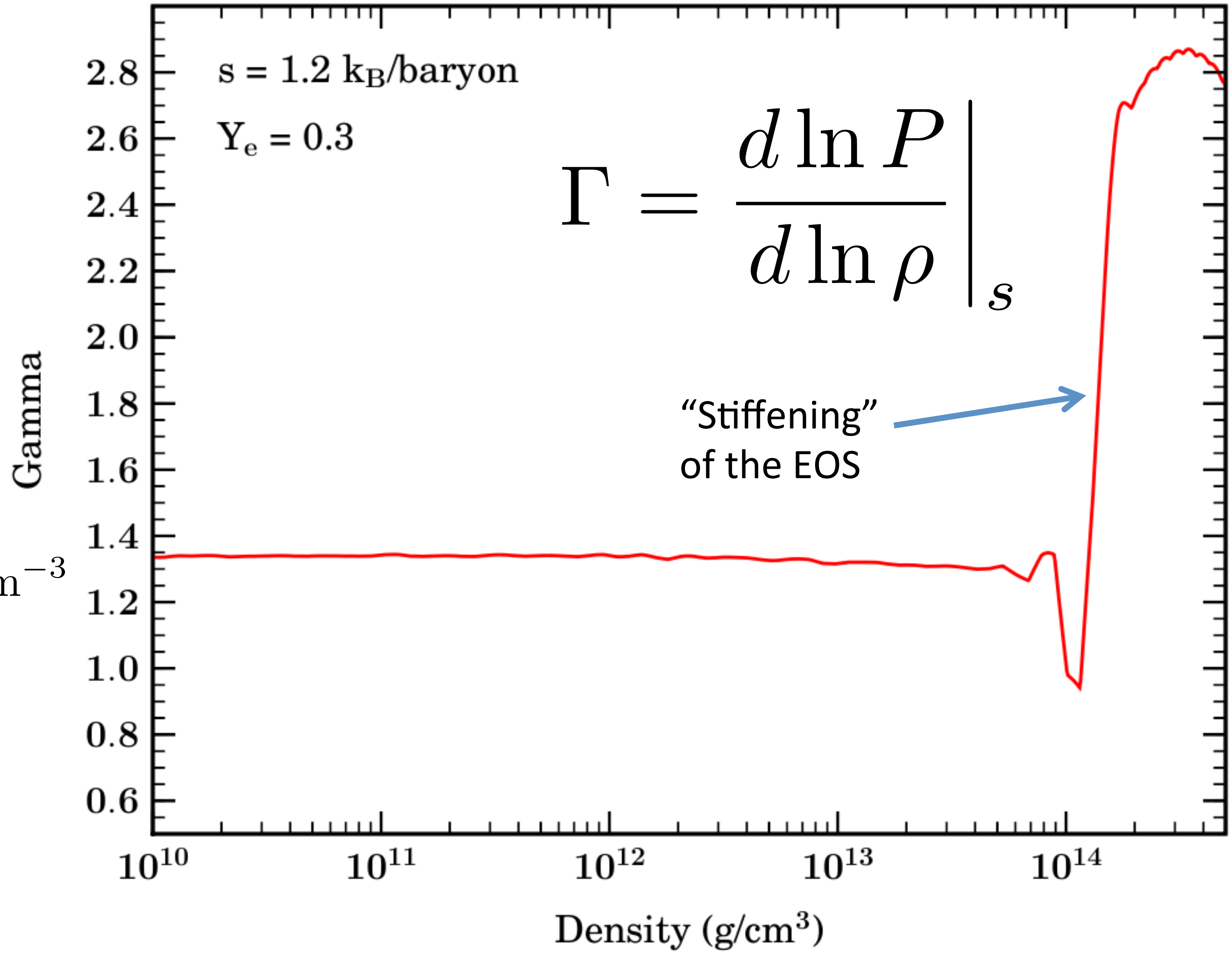
$$r_0 = 1.25\,\mathrm{fm}$$

Nuclear Density:

$$\bar{\rho}_{\rm nuc} = \frac{A\,m_b}{\frac{4}{3}\pi R_{\rm nuc}^3}$$

$$\rho_{\rm nuc} \sim 2.7 \times 10^{14}\,\mathrm{g\,cm^{-3}}$$

$$n_{\rm nuc} \sim 0.16\,\mathrm{fm}^{-3}$$



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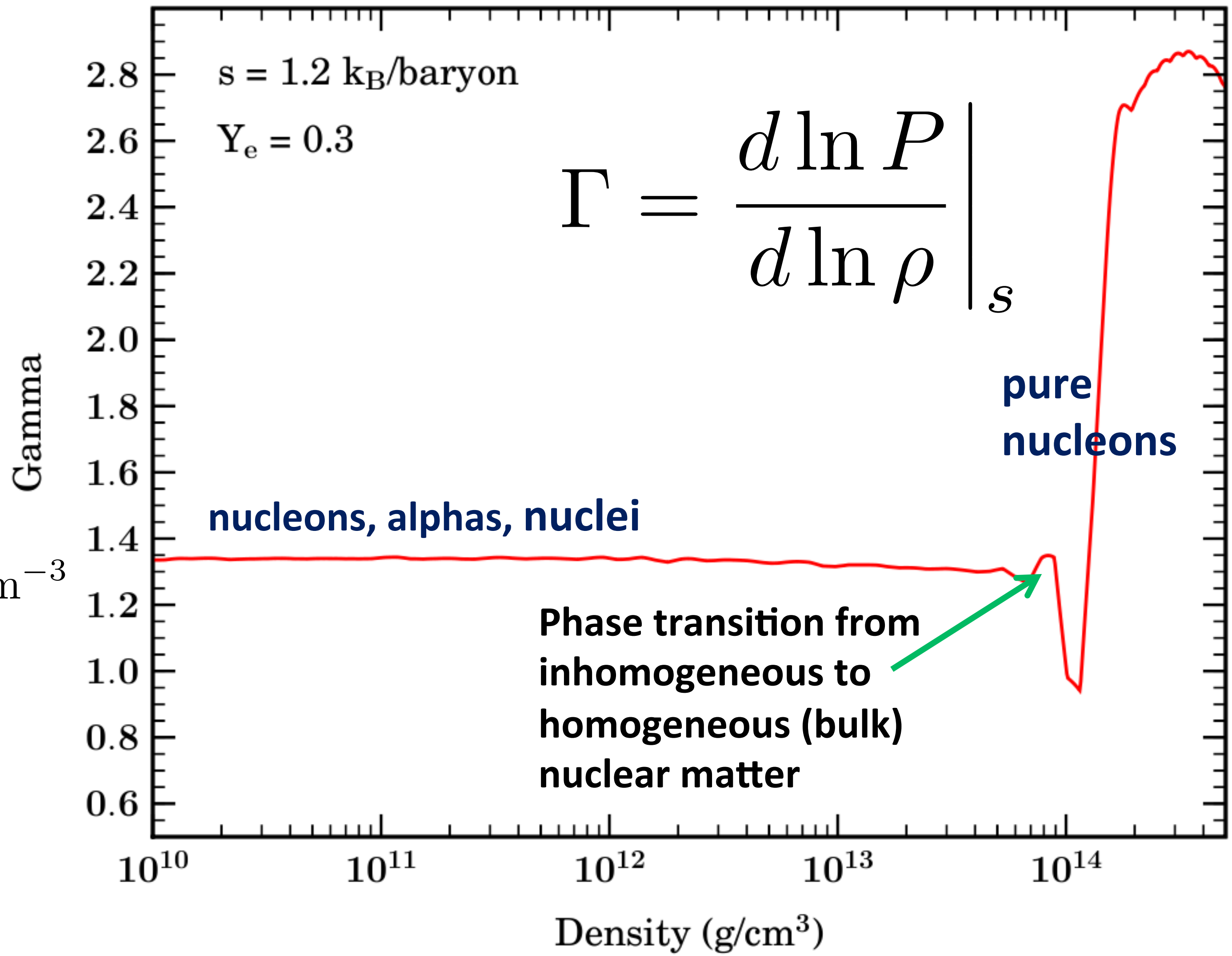
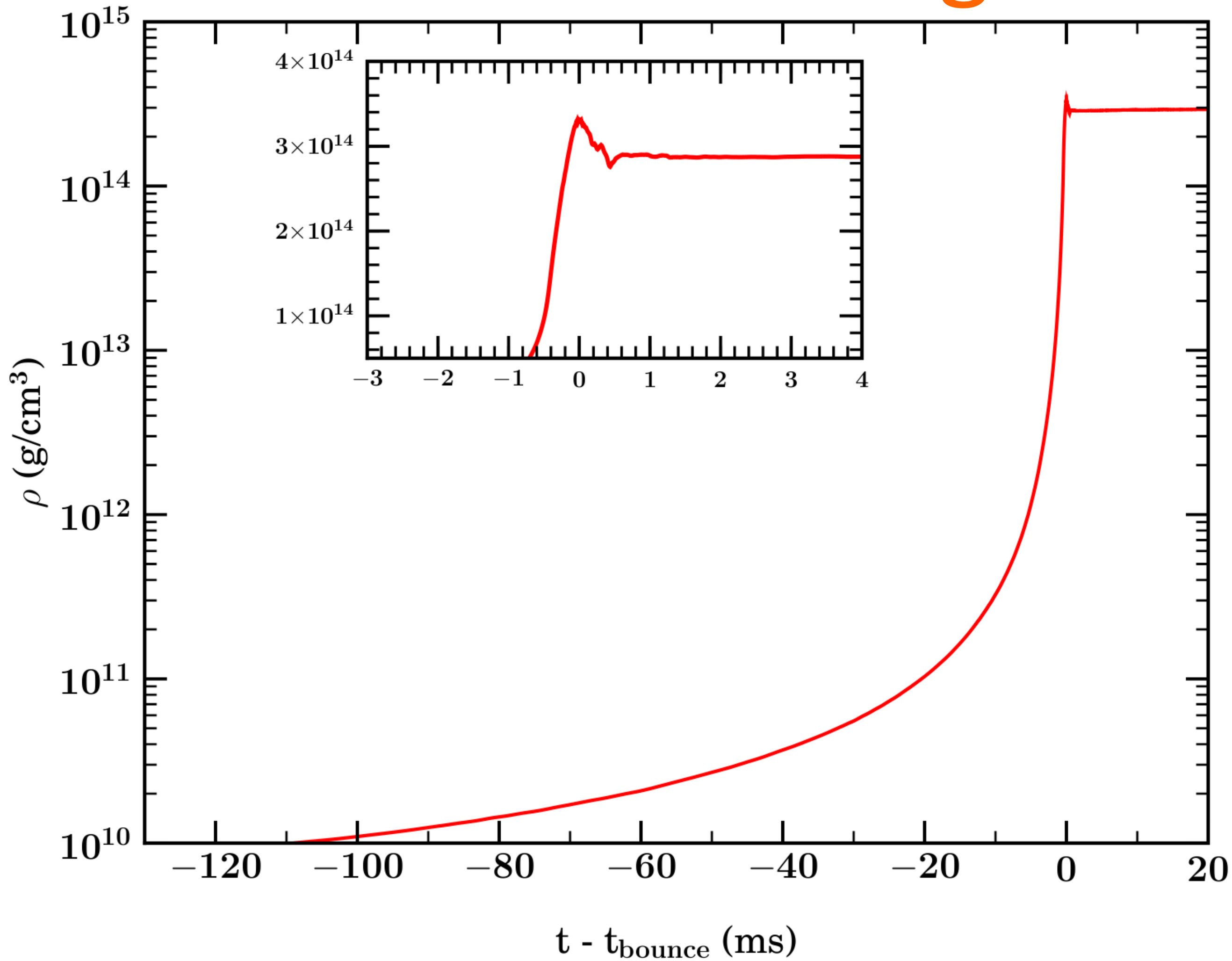


What is causing the stiffening of the nuclear EOS?

Effect of Stiffening



- **Inner Core** reaches ρ_{nuc}, rebounds ("bounces") **into still infalling outer core**.

Nuclear Equation of State

First some thermodynamics:

First Law $$dQ = TdS = dE + PdV - \sum_i \mu_i dN_i$$

In specific quantities per particle (baryon):

$$d\epsilon = -Pd\left(\frac{1}{n}\right) + Tds + \sum_i \mu_i d\left(\frac{n_i}{n}\right)$$

$$\epsilon = \epsilon(n, s, \{Y_i\})$$ but NSE: $$\epsilon = \epsilon(n, s, Y_e)$$

Prefer to work in $X = X(n, T, Y_e)$

$$n = \frac{N}{V}$$

$$dV = \frac{1}{N} d\left(\frac{1}{n}\right)$$

$$\frac{n_i}{n} = Y_i$$

Helmholtz free energy:

$$f = f(n, T, Y_e) = \epsilon - Ts$$

At fixed T, n, and composition, Helmholtz Free Energy is minimized in equilibrium.

Nuclear Equation of State

EOS from the Free Energy:

$$f = f(n, T, Y_e) = \epsilon - Ts$$

$$df = -Pd\left(\frac{1}{n}\right) + sdT + \sum_i \mu_i d\left(\frac{n_i}{n}\right) \qquad \frac{d}{d(\frac{1}{n})} = -n^2 \frac{d}{dn}$$

Obtain thermodynamic quantities via derivatives of f:

$$P = n^2 \frac{\partial f}{\partial n}\bigg|_{T,Y_e} \qquad s = -\frac{\partial f}{\partial T}\bigg|_{n,Y_e} \qquad \mu_i = \frac{\partial f}{\partial n_i}\bigg|_{n,T}$$

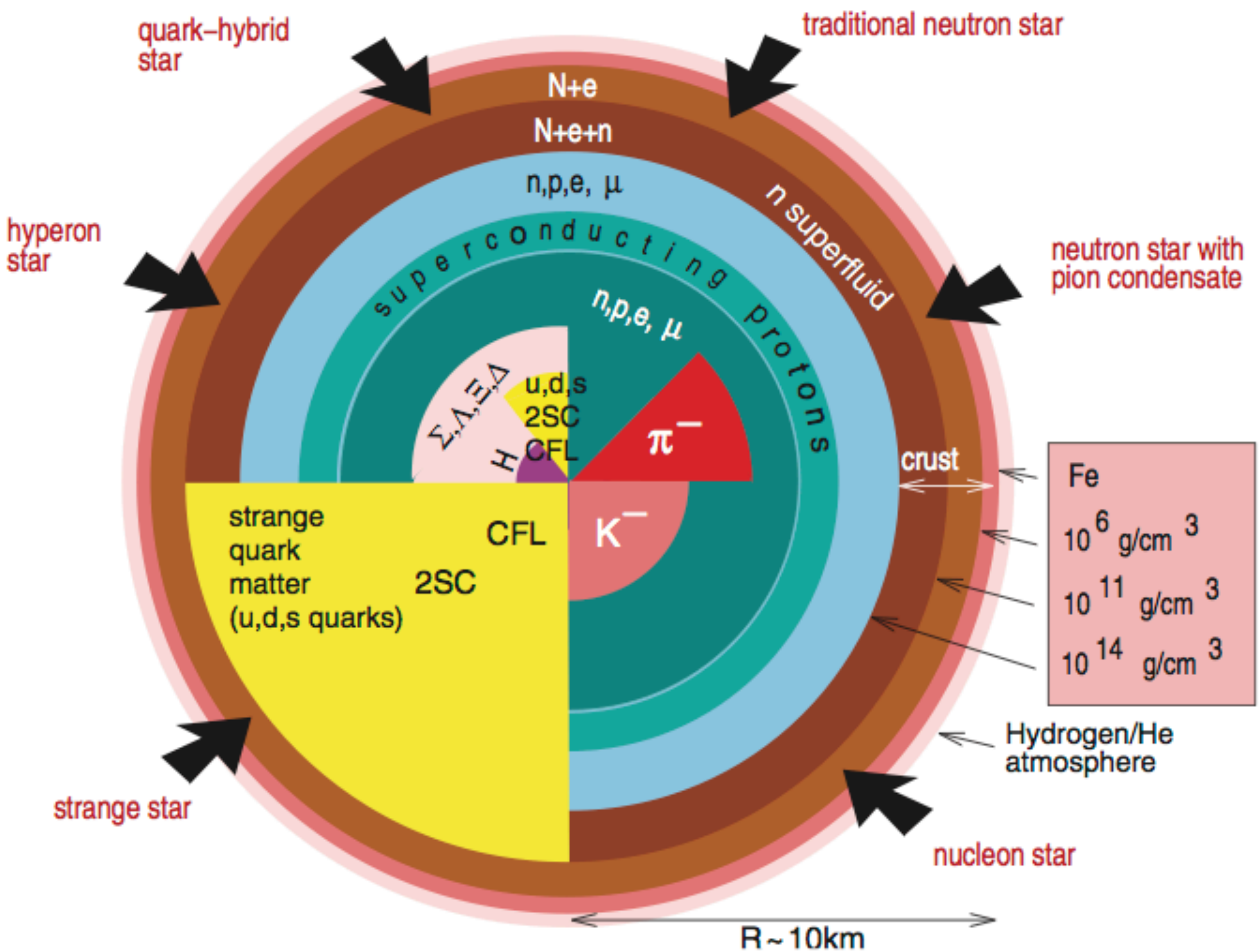
Finding the EOS = min(f) for a given n, T, Y_e. This also fixes mass fractions of constituent particles.

Typical constituents: n, p, α, representative nucleus with (A,Z) or NSE ensemble {A_i, Z_i}.
At high densities: exotica such as hyperons, kaons, etc.

Generally: $f = f_{\text{baryon}} + f_e + f_\gamma$ (electrons, photons independent of baryons)

Nuclear Equation of State

- Simplification: T=0, pure neutron & proton gas. Appropriate (?) for interior of cold neutron stars.



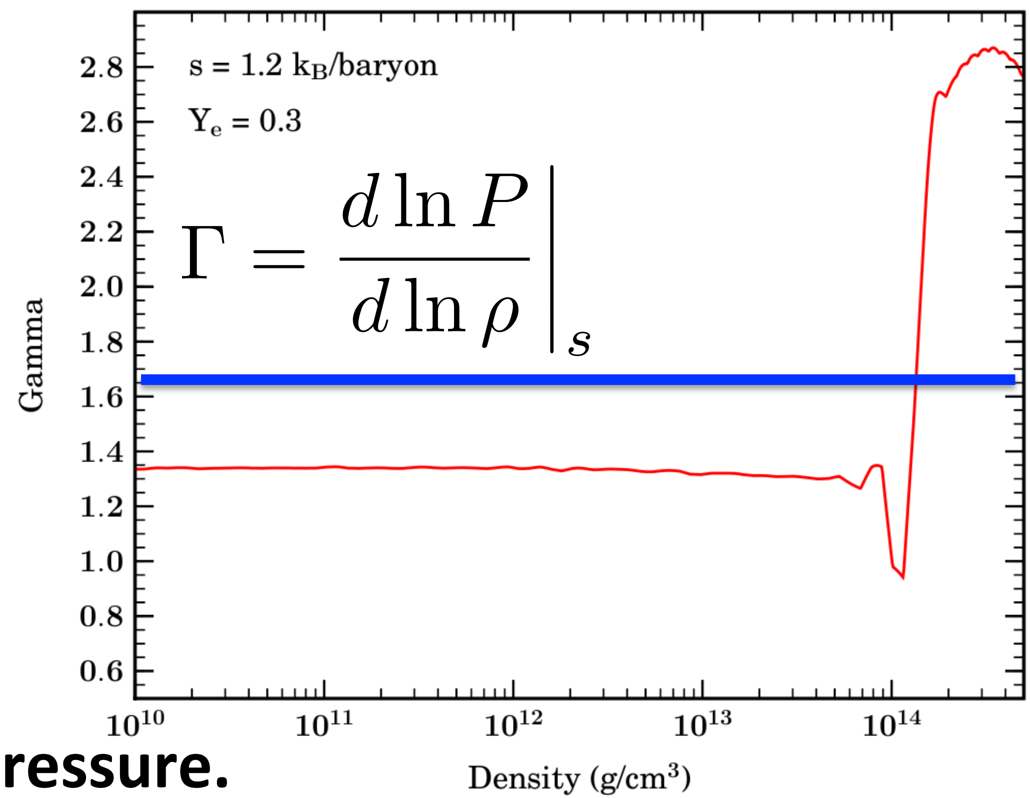
Nuclear Equation of State

- T=0, pure neutron & proton gas. $f = \epsilon$

$$\epsilon(n_n, n_p) = \frac{3}{5}\frac{p_{F,n}^2}{2m_n}\frac{n_n}{n} + \frac{3}{5}\frac{p_{F,p}^2}{2m_p}\frac{n_p}{n} \quad p_F = (3\pi^2\hbar^3)^{1/3} n^{1/3}$$

$$P = n^2 \frac{\partial \epsilon}{\partial n} \propto n^{5/3}$$

$$\Gamma = \left.\frac{d\ln P}{d\ln \rho}\right|_s = \frac{5}{3}$$



Not sufficiently stiff!
What is missing?

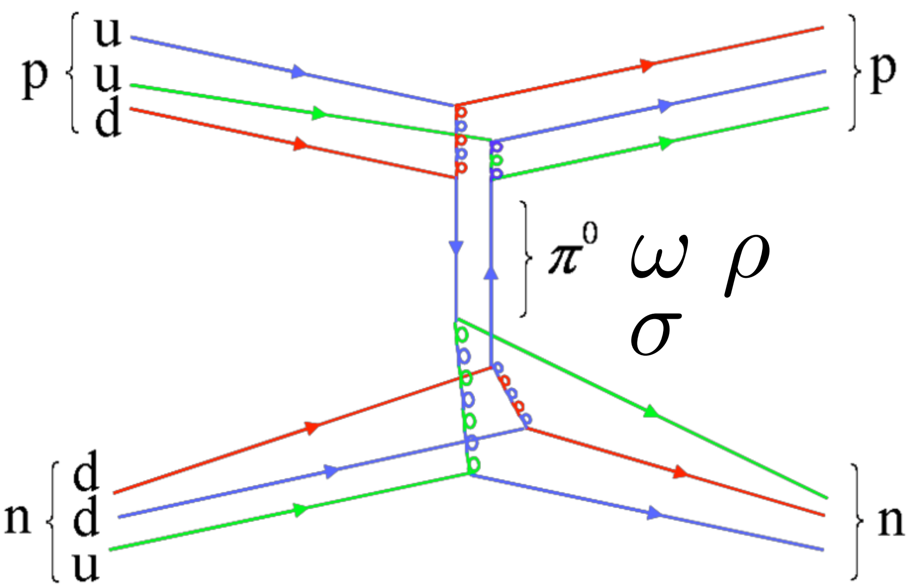
Neutron stars are not supported by degeneracy pressure.

Nuclear Equation of State

- T=0, interacting pure neutron & proton gas.

$$\epsilon(n_n, n_p) = \frac{3}{5}\frac{p_{F,n}^2}{2m_n}\frac{n_n}{n} + \frac{3}{5}\frac{p_{F,p}^2}{2m_p}\frac{n_p}{n} + \frac{V_{np}(n_n, n_p)}{n}$$

nucleon-nucleon (NN) potential energy density



- Nuclear force is NN many-body interaction = "effective" strong force interaction.
- Mediated by mesons:
  π (s=0), σ (s=0), ω (s=1), ρ (s=1)
- Dependent on separation and spin orientation. **Scalar**, **vector**, and **tensor** components.
  Vector component is repulsive.

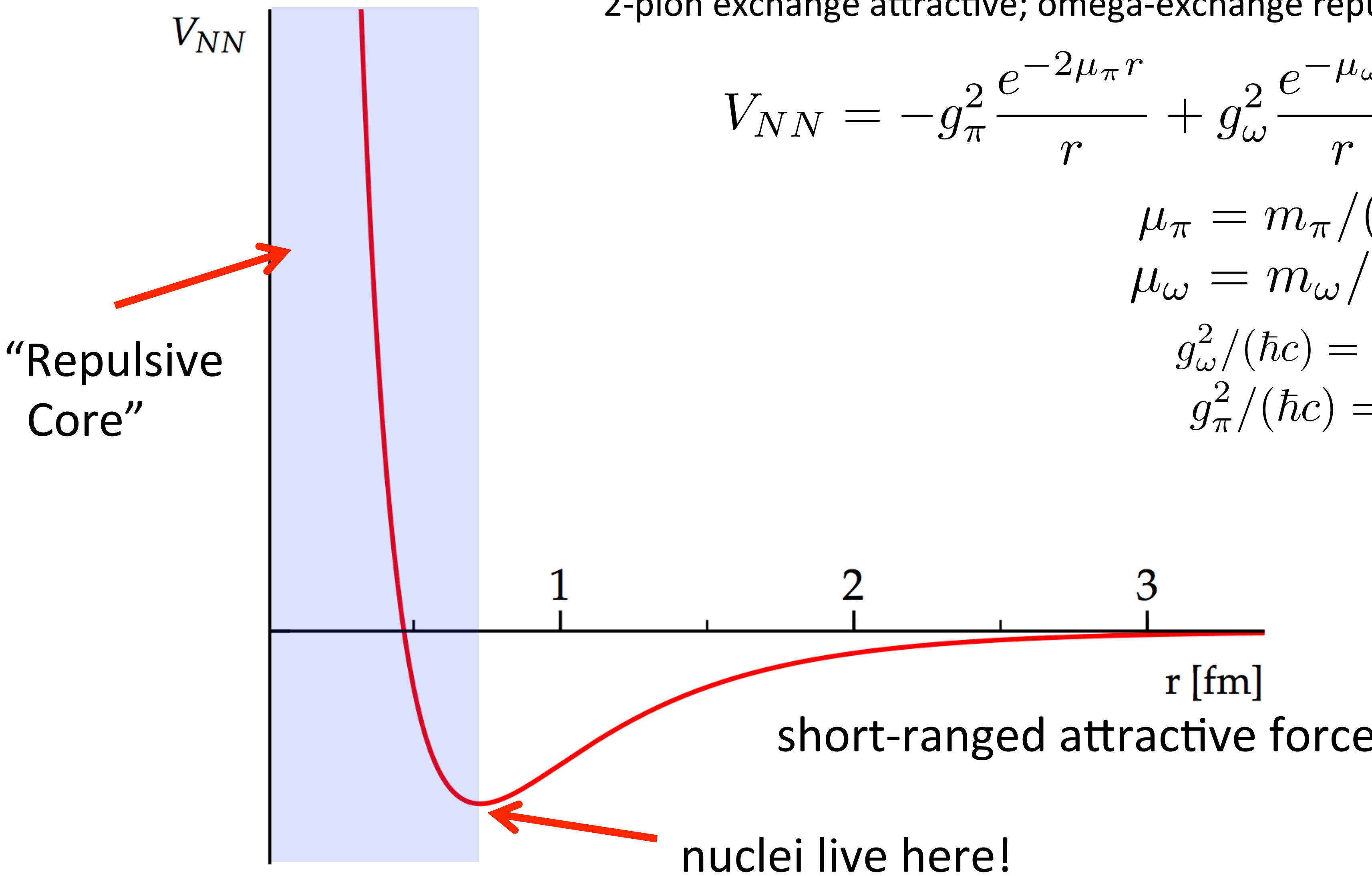
Nucleon-Nucleon Interaction

Example: Bethe & Johnson 74

2-pion exchange attractive; omega-exchange repulsive

$$V_{NN} = -g_\pi^2 \frac{e^{-2\mu_\pi r}}{r} + g_\omega^2 \frac{e^{-\mu_\omega r}}{r}$$

$$\mu_\pi = m_\pi/(\hbar c)$$
$$\mu_\omega = m_\omega/(\hbar c)$$
$$g_\omega^2/(\hbar c) = 29.6$$
$$g_\pi^2/(\hbar c) = 10$$



Obtaining an EOS

- Brute force: Solve quantum many-body interactions with V_{NN} (e.g. via Hartree-Fock approach).

- Mean field approximation (write down Lagrangian for nucleons moving in effective meson fields), introduce parameters to match laboratory nuclei or observations.

- Phenomenological approach: Liquid drop model with parameters from theory (V_{NN}), experiments, and observations.

Liquid Drop Model

Bethe & von Weizsäcker 1935/37

Nuclear masses:

$$M(N,Z) = Zm_p + Nm_n - BE$$

$$BE = a_V A - a_\sigma A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_{\rm sym} \frac{(N-Z)^2}{A} + \delta(N,Z)$$

Volume term — Surface term — Coloumb term — Symmetry Term — Pairing Term

$$a_V \simeq 16\,\mathrm{MeV} \quad a_\sigma \simeq 18\,\mathrm{MeV} \quad a_C \simeq 0.7\,\mathrm{MeV} \quad a_{\rm sym} \simeq 23\,\mathrm{MeV}$$

$$\delta(N,Z) = \begin{cases} -\delta_0 & Z, N\ \mathrm{even} \\ 0 & Z+N\ \mathrm{odd} \\ \delta_0 & Z, N\ \mathrm{odd} \end{cases} \qquad \delta_0 = \frac{a_P}{A^{1/2}} \qquad a_P \simeq 12\,\mathrm{MeV}$$

Liquid Drop Model -> EOS

(e.g. Lattimer & Swesty 1991, Lattimer & Prakash 2007, Lattimer & Lim 2013)

- Near nuclear saturation density $n_s \sim 0.16\,\mathrm{fm}^{-3}$, expand energy per baryon:

$$\epsilon(n,x) = -16\,\mathrm{MeV} + \frac{1}{18}K\left(1-\frac{n}{n_s}\right)^2 + \frac{K'}{27}\left(1-\frac{n}{n_s}\right)^3 + E_{\mathrm{sym}}(n)(1-2x)^2 + \ldots$$

At T=0: $f = \epsilon$ $\qquad x = Y_p = Y_e$

$$\left.\begin{array}{ll} K \simeq 240\,\mathrm{MeV} & \text{incompressibility} \\ E_{\mathrm{Sym}}(n_s) = S_v \approx 29.0 - 32.7\,\mathrm{MeV} & \text{symmetry energy} \\ K' \approx 1780 - 2380\,\mathrm{MeV} & \text{skewness} \end{array}\right\}$$

experimental & astrophysical constraints

- Write out energy of bulk nuclear matter according to nuclear force model (e.g., Skyrme 1959) and use T=0, n=ns, and above expansion to set parameters of nuclear force.
- Introduce model for nuclei & alpha particles, then minimize f.

Neutron Star Structure

Newtonian: $$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \qquad \frac{dM}{dr} = 4\pi r^2 \rho$$ (no maximum mass!)

GR:

$$\frac{dP}{dr} = -G(\rho(1+\epsilon/c^2) + P/c^2)\frac{M + 4\pi r^3 p/c^2}{r(r - 2GM/c^2)}$$

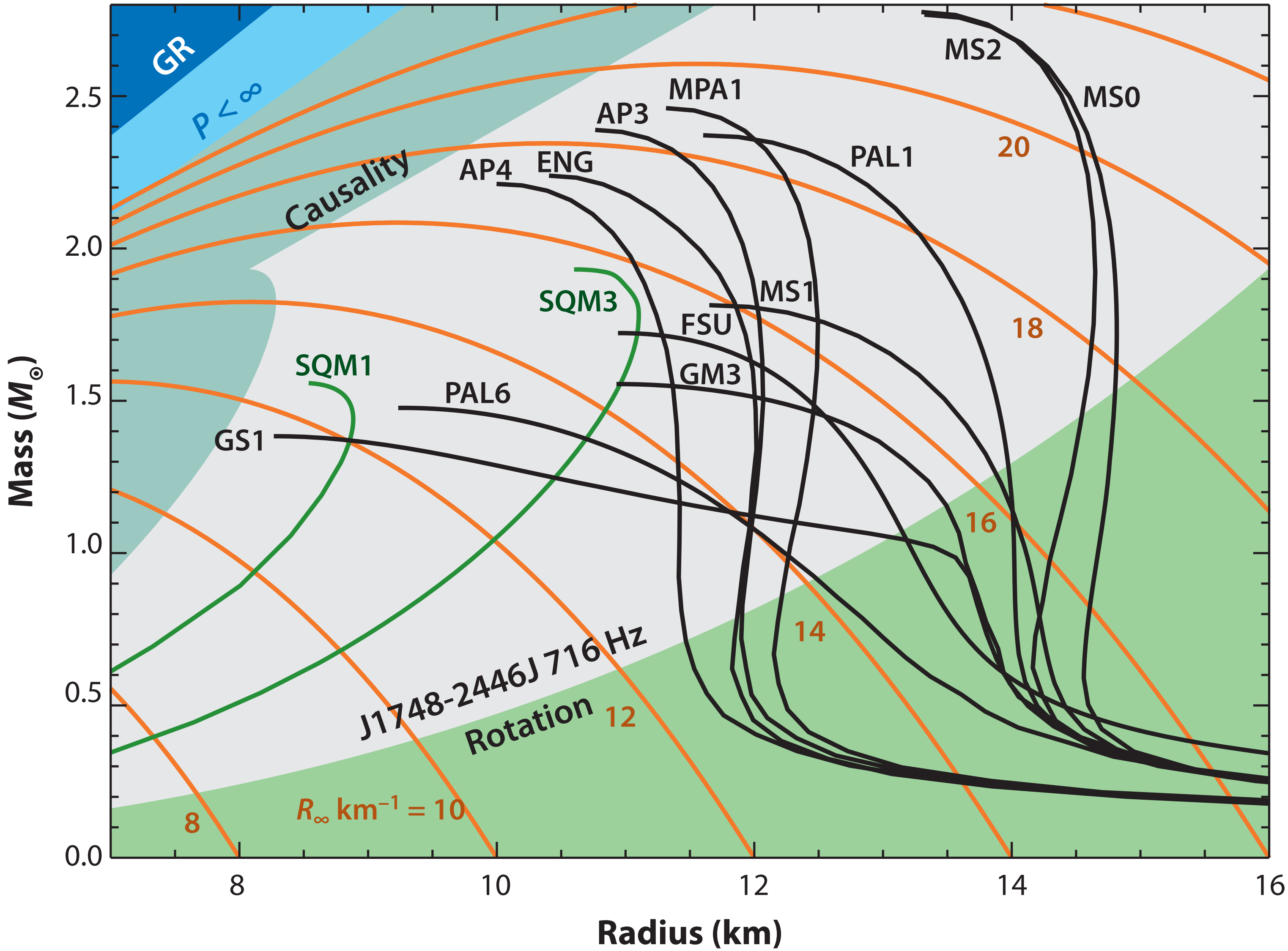
$$\frac{dM_g}{dr} = 4\pi r^2 \rho(1+\epsilon/c^2)$$ gravitational mass

$$\frac{dM_b}{dr} = \frac{4\pi r^2}{\sqrt{1-\frac{2GM}{rc^2}}}\rho$$ baryonic mass

Radius is circumferential radius!

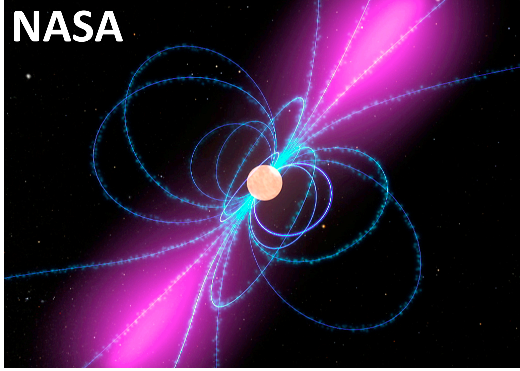
- Solve by ODE integration from r=0, invert P(ρ) at each step to obtain ρ.

EOS & Neutron Star Structure



Knowing masses and radii would really help!!!

Neutron Star Masses



Pulsar with binary companion:
Mass function for pulsar

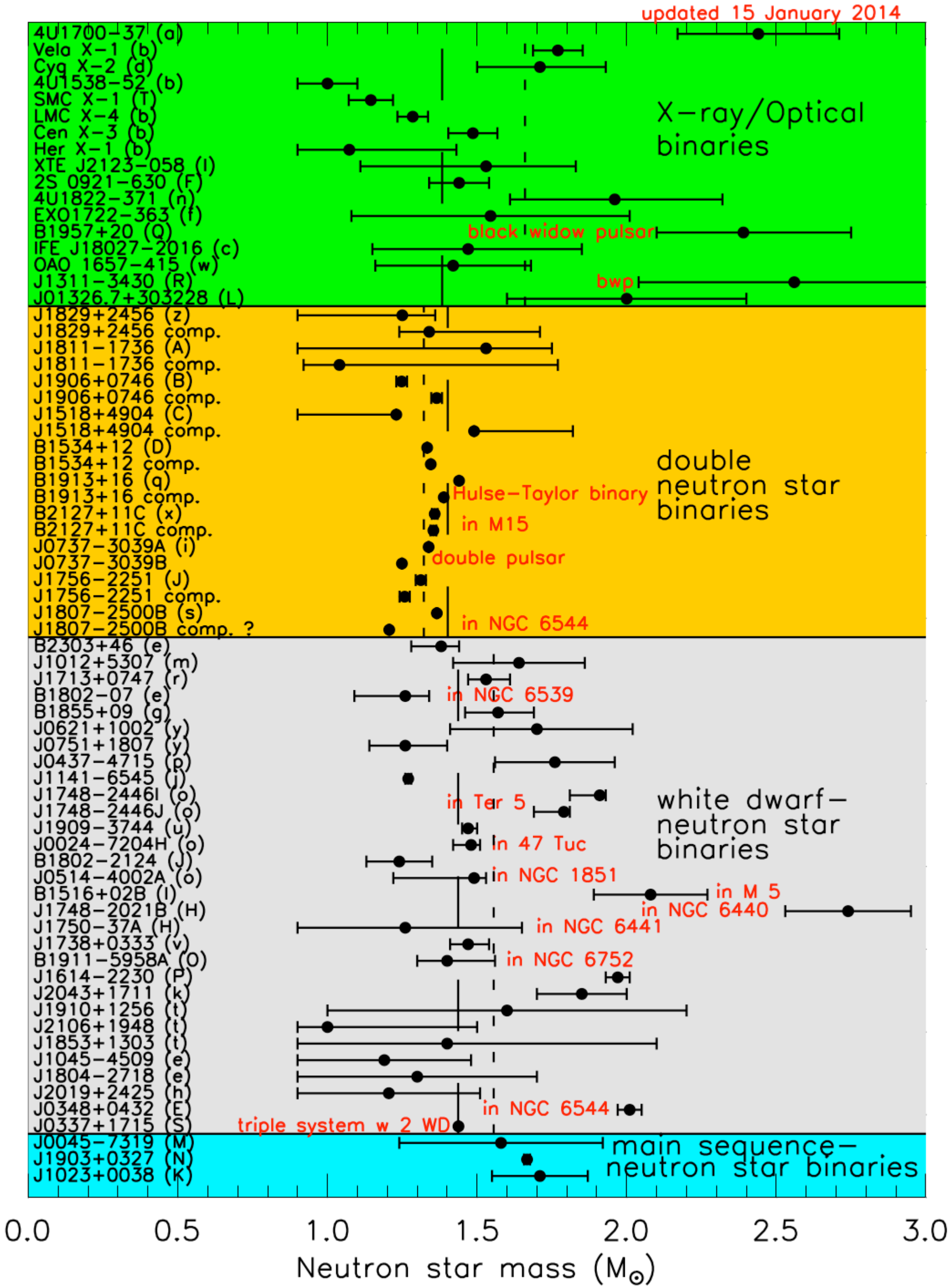
Companion mass

$$f_p = \left(\frac{2\pi}{P}\right)^2 \frac{(a_p \sin i)^3}{G} = \frac{(M_c \sin i)^3}{M^2}$$

Orbital inclination

Total system mass

- Must know/infer **companion mass** and **inclination** to get M_P.
- Different kinds of binaries:
  X-ray binaries (accreting NSs), double NS binaries,
  NS–normal-star binaries, NS–WD binaries.
- Companion mass: via stellar models or relativistic effects.
- Inclination: most difficult. In relativistic binaries:
  Shapiro time delay (delay of pulsar pulses by gravity of companion)



Lattimer 14,
http://stellarcollapse.org

X-ray binaries

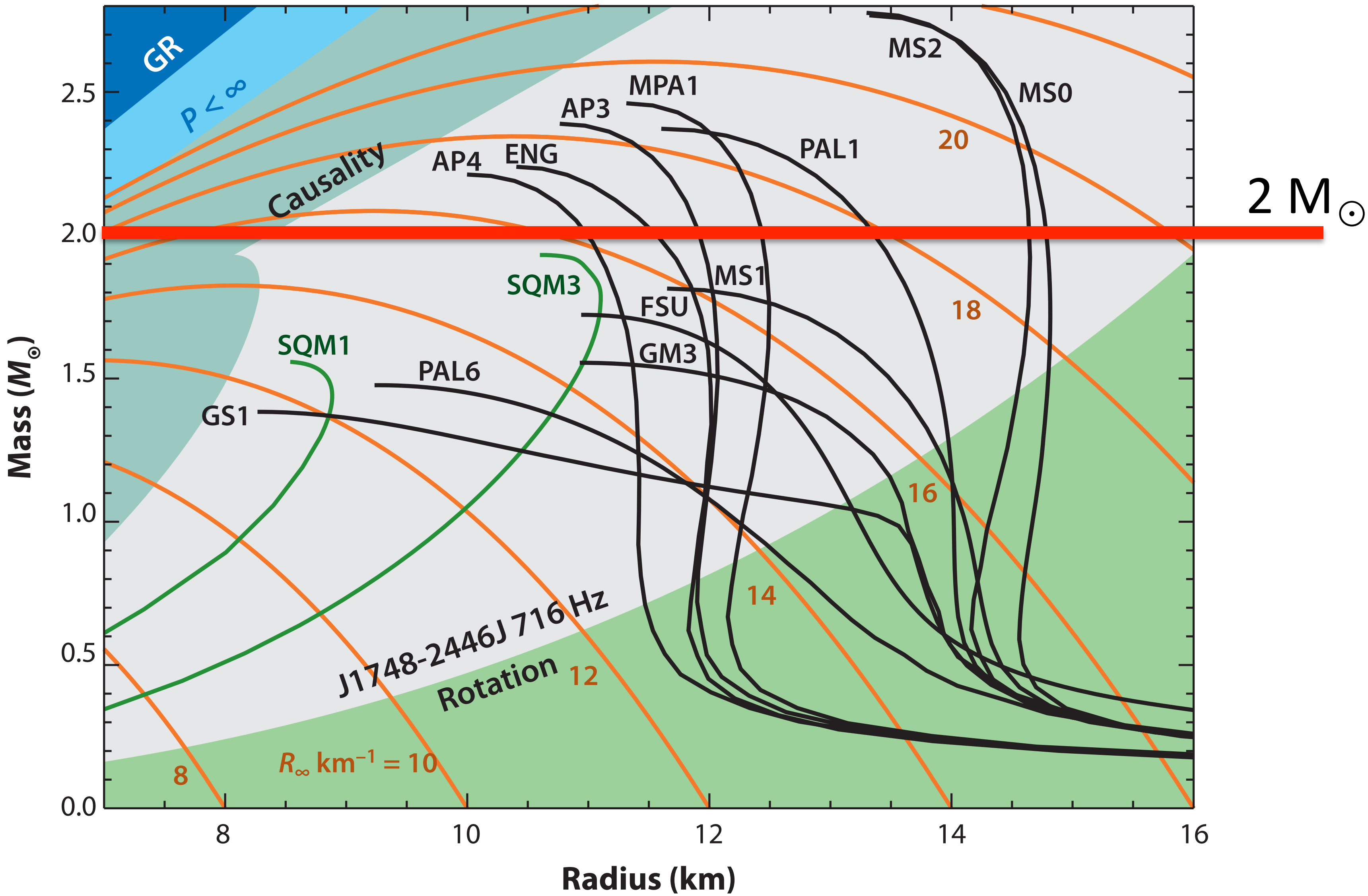
NS+NS

WD+NS

NS + normal star

Most massive:
PSR J1614-2230
1.97+-0.04 $M_\odot$
PSR J0348+0432
2.01+-0.04 $M_\odot$

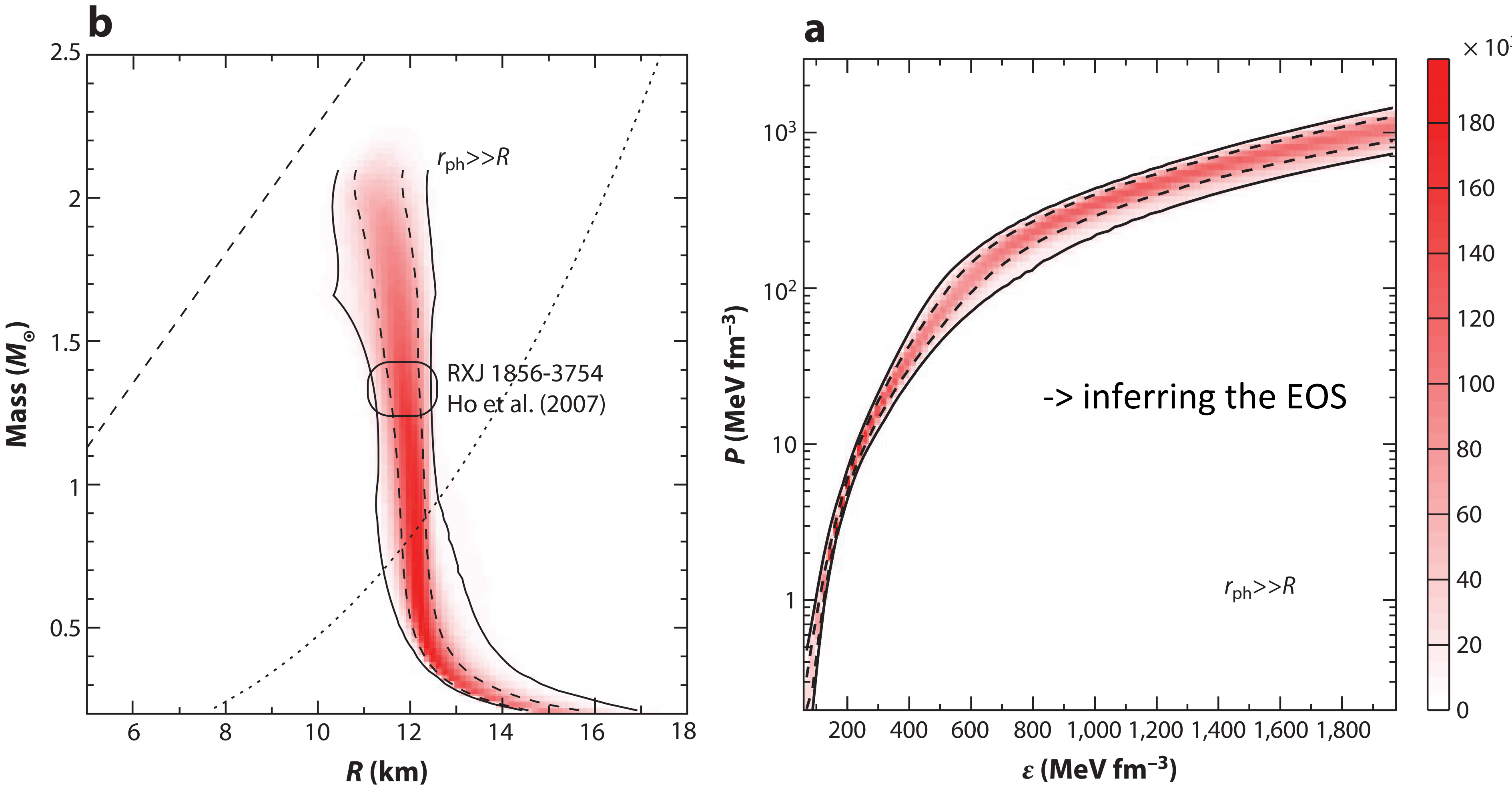
Neutron Star Structure & EOS Constraints



Neutron Star Radii

- So far no robust NS radius (or mass&radius) measurements.
- Main approaches: (see Lattimer 2012)
  - X-ray observation of quiescent and bursting NSs in galactic X-ray binaries.
  - GW signal from tidal deformation and disruption of NS in BHNS merger.
  - GW signal from tidal deformation and postmerger oscillations in NSNS merger.
  - Neutrino signal from the proto-NS in the next galactic CCSN.

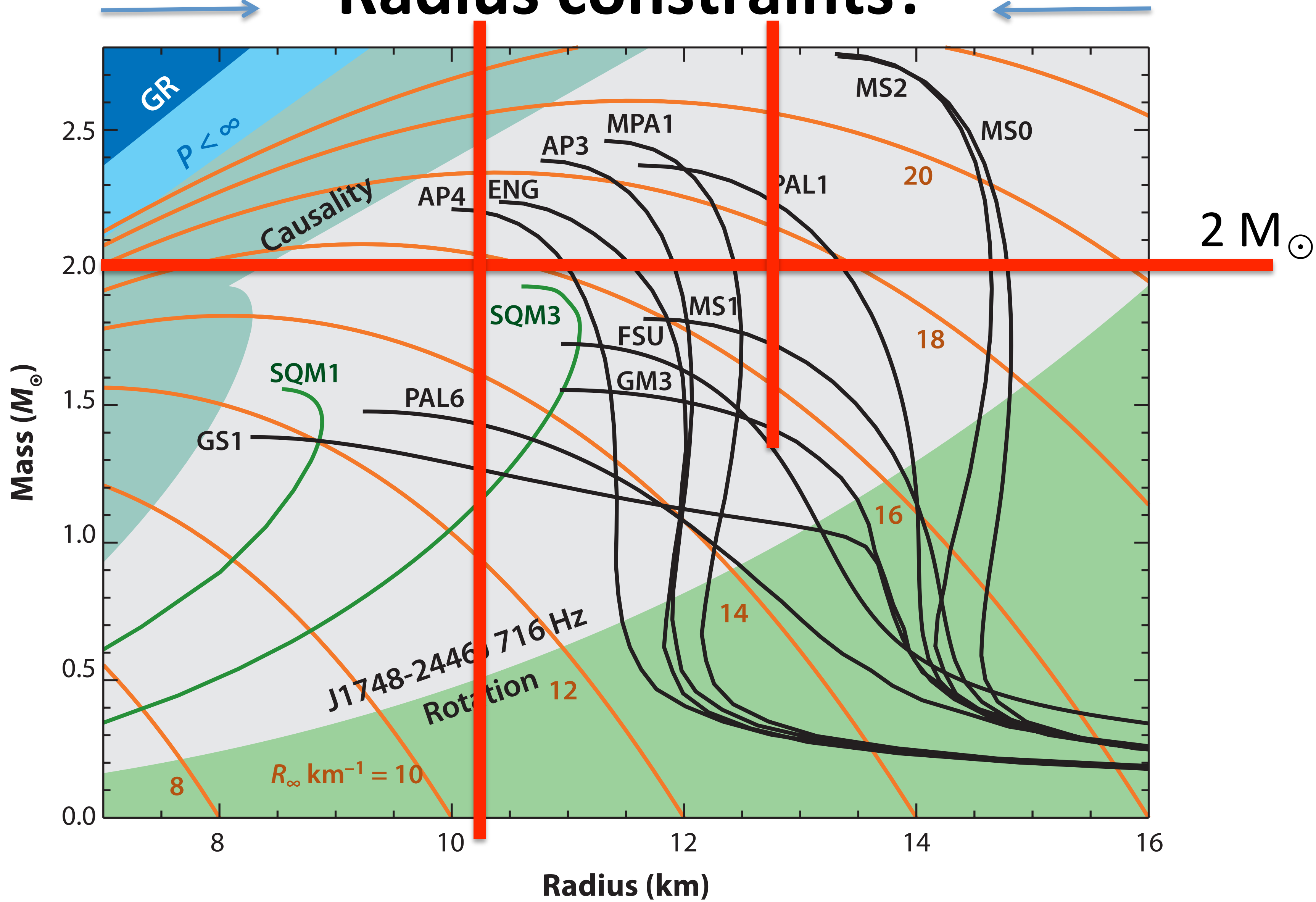
Neutron Star Masses & Radii



Statistical Analysis of observational data: Steiner+10,+12, Lattimer 12

Warning: Does not fix model dependence of M, R estimates!

Radius constraints?



Literature on the Nuclear EOS

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- Haensel, Pothekhin, Yakhovlev, Neutron Stars – Equation of State and Structure, Springer, 2007
- Bethe 1990, Rev. Mod. Phys. 62, 801
- Bethe, Brown, Applegate, Lattimer 1979, Nuc. Phys. A 324, 487
- Lattimer, Pethick, Ravenhall 1985, Nuc. Phys. A 432, 646
- Lattimer & Swesty 1991, Nuc. Phys. A 535, 331
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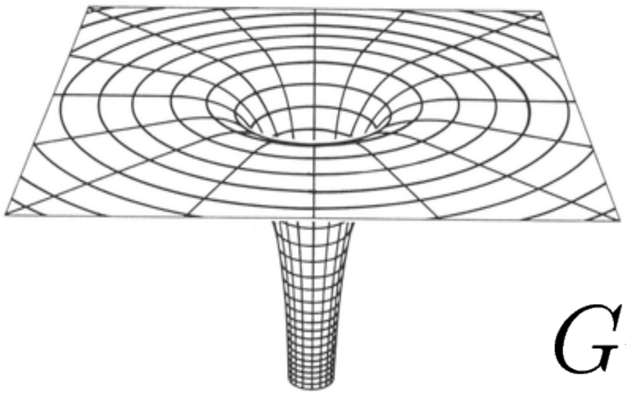
Numerical Relativity, General-Relativistic Hydrodynamics and the Einstein Toolkit

General Relativity

http://ion.uwinnipeg.ca/~vincent/4500.6-001/Cosmology/embedding.gif



Will soon set:

$$G = c = M_\odot = 1$$

General Relativity:
A geometric theory of gravity

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Einstein Tensor:
"Geometry"
"Curvature"
(symmetric)

Stress-Energy Tensor:
mass/energy/momentum/
pressure/stress densities
sourcing curvature.
(symmetric)

Ten independent tensor components in 4D spacetime

Metric and Derivatives

- Metric $g_{\mu\nu}$ is fundamental concept in GR. Used to measure physical distances in curved spacetime. Can derive "curvature" from metric.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

(Note: Einstein sum convention)

- Partial derivatives $\frac{\partial}{\partial x^\alpha} X = \partial_\alpha X = X_{,\alpha}$ are coordinate dependent.

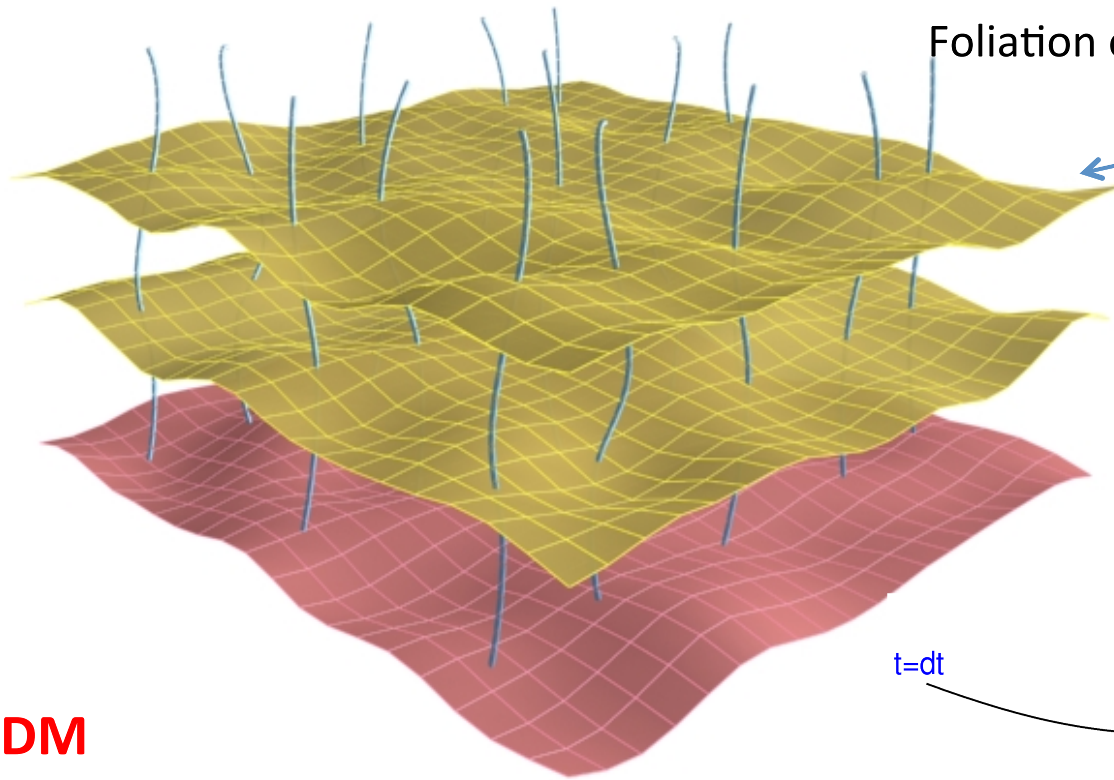
- Covariant derivatives are coordinate independent:

$$D_\beta X^\alpha = X^\alpha{}_{;\beta} = \partial_\beta X^\alpha + \Gamma^\alpha{}_{\beta\gamma} X^\gamma$$

"Christoffel Symbol" contains first derivatives of metric.

- Any physically meaningful theory must be covariant (coordinate independent).

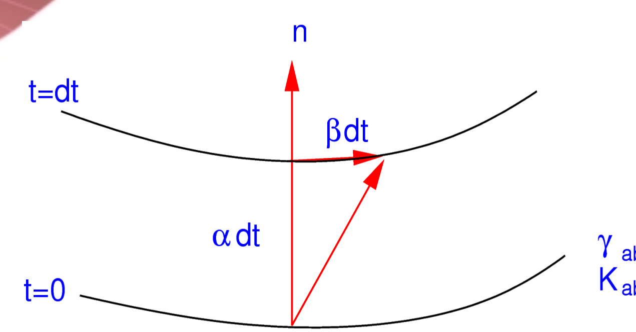
Basic Idea of Numerical Relativity



ADM

3+1 split of spacetime

Figure: C. Reisswig



$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

- 12 first-order hyperbolic *evolution* equations.
- 4 elliptic *constraint* equations
- 4 coordinate gauge degrees of freedom: α, β^i.

3+1 Split

3+1 split – key objects:

$$\begin{aligned} g_{\mu\nu} &= 4-\text{metric} \\ \gamma_{ij} &= 3-\text{metric} \\ \alpha &= \text{lapse function} \\ \beta^i &= \text{shift vector} \end{aligned}$$

$$\gamma = \det(\gamma_{ik})$$

$$\sqrt{-g} = \alpha\sqrt{\gamma}$$

$$g_{00} = -\alpha^2 + \beta_i\beta^i \quad g_{0i} = -\gamma_{ij}\beta^j \quad g_{ij} = \gamma_{ij}$$

Extrinsic curvature: ≈ time derivative of 3-metric

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

4-velocity: $u^\mu\ ;\ u^\mu u_\mu = -1\ ;\ u^\mu = (-1,0,0,0)$ in rest frame

3-velocity: Eulerian observer moving along time-like normal n^μ.

$$v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}$$

$$W = (1 - v^i v_i)^{-1/2}$$

$$u^0 = \frac{W}{\alpha} \quad u^i = W\left(v^i - \frac{\beta^i}{\alpha}\right)$$

ADM Equations

(Historic: Arnowitt-Deser-Misner; York)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha \Big[R_{ij} K K_{ij} - 2K_i m K^m_j$$

Ricci tensor

$$-8\pi \left(S_{ij} - \frac{1}{2}\gamma_{ij} S \right) - 4\pi \rho_{\mathrm{ADM}} \gamma_{ij} \Big]$$

$$+\beta^m D_m K_{ij} + K_{im} D_j \beta^m + K_{mj} D_i \beta^m$$

$$S^i = -\gamma^{i\mu} n^\nu T_{\mu\nu} \qquad \rho_{\mathrm{ADM}} = n_\mu n_\nu T^{\mu\nu}$$

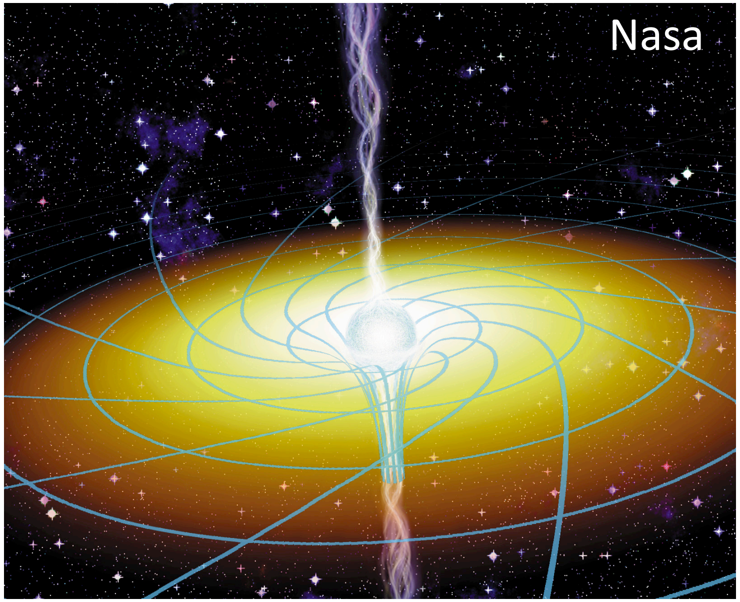
$$S_{ij} = \gamma_{i\mu}\gamma_{j\nu} T^{\mu\nu} \qquad S, K - \text{traces of } S_{ij}, K_{ij}$$

Constraints:

Hamiltonian	$R + K^2 - K_{ij}K^{ij} - 16\pi\rho_{\mathrm{ADM}} = 0$
Momentum	$D_j K^{ij} - \gamma^{ij} D_j K - 8\pi S^i = 0$

Practical Numerical Relativity

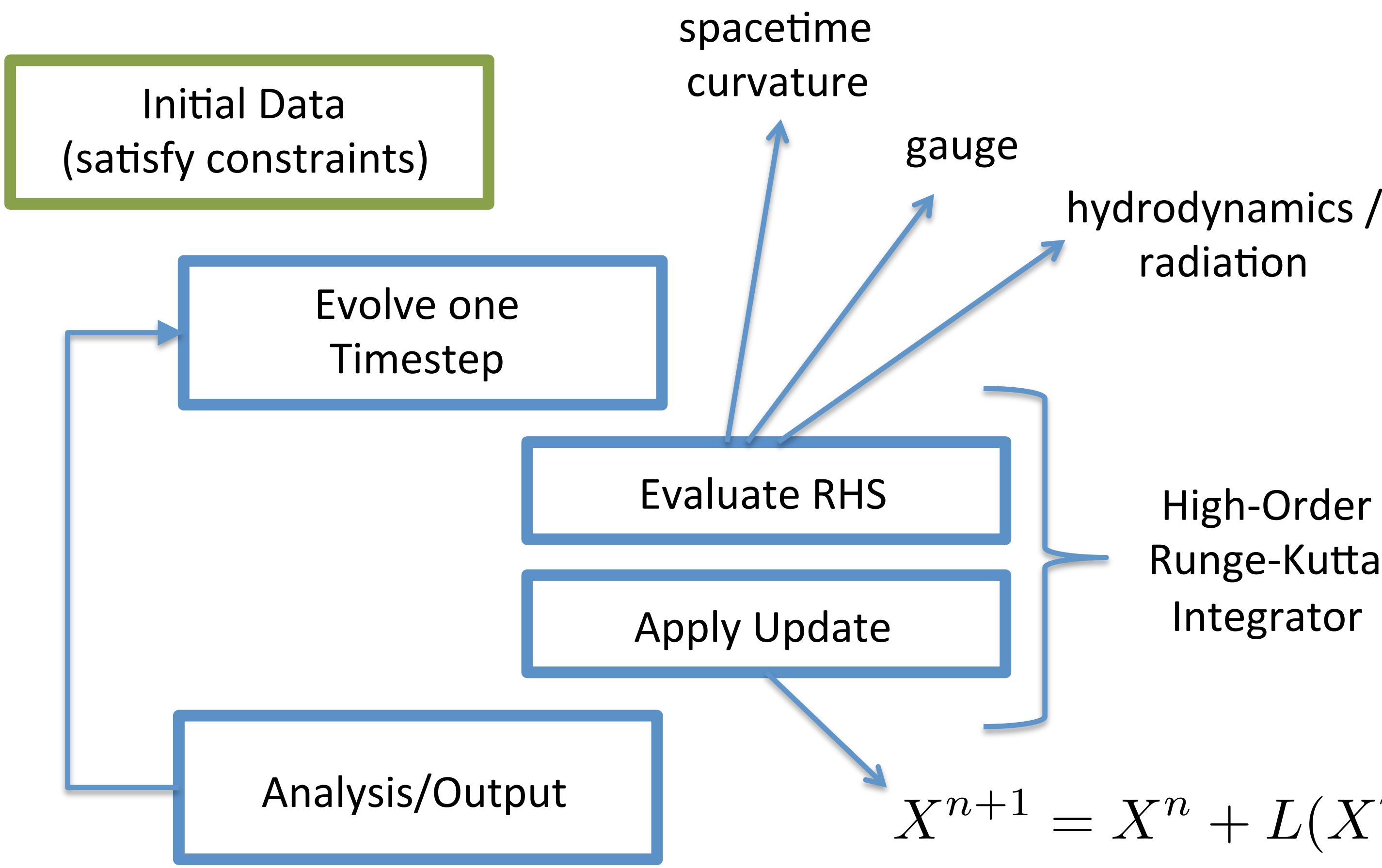
- Have not yet specified gauge conditions: Anything goes?



- GR dynamics will twist, squeeze, stretch coordinates.
- GR can develop coordinate singularities and physical singularities.
- For numerically stable evolution, must avoid singularities and control coordinate distortion.

- ADM form of the Einstein equation is unstable!
  -> small errors in constraints get amplified over time!
- In practice, must use different (and stable) formulation (good ones available since early 2000s): BSSN, Z4c, Generalized Harmonic.
- Finite differences or spectral methods for discretization.

Schematic Numerical Relativity Simulation



$$X^{n+1} = X^n + L(X^n)\Delta t$$

(first order)

$$\Delta t < \Delta x / c$$

Complication: Adaptive Mesh Refinement

GR Hydrodynamics

(neglecting magnetic fields)

$$j^\mu = \rho u^\mu$$ mass flux

$$T^{\mu\nu} = \rho h u^\mu u^\nu - P g^{\mu\nu}$$

Stress Energy Tensor of an ideal fluid (inviscid, no magnetic field)

$$h = 1 + \epsilon + P/\rho$$

relativistic specific enthalpy

Conservation of mass, momentum, and energy:

$$j^\mu{}_{;\mu} = (\rho u^\mu)_{;\mu} = 0$$ Mass conservation

$$T^{\mu\nu}{}_{;\mu} = (\rho h u^\mu u^\nu - P g^{\mu\nu})_{;\mu} = 0$$ Energy-momentum conservation

covariant derivative; here: divergence takes into account that space is curved.

GR Hydrodynamics

$$\gamma = \det(\gamma_{ik})$$

$$\sqrt{-g} = \alpha\sqrt{\gamma}$$

Flux-conservative Formulation:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^{\,i}}{\partial x^{\,i}} = \mathbf{S}$$

$$W = (1 - v^i v_i)^{-1/2}$$

$$\mathbf{U} = [\hat{D}, \hat{S}_j, \hat{\tau}] \quad \hat{D} = \sqrt{\gamma}\rho W,$$ conserved mass

$$\hat{S}^i = \sqrt{\gamma}\rho h W^{\,2} v^{\,i},$$ conserved momenta

$$\hat{\tau} = \sqrt{\gamma}\left(\rho h W^{\,2} - P\right) - D$$ conserved energy

$$\mathbf{F}^{\,i} = \alpha\left[\hat{D}\tilde{v}^{\,i}, \hat{S}_j\tilde{v}^{\,i} + \delta^{\,i}_{j}P, \hat{\tau}\tilde{v}^{\,i} + Pv^{\,i}\right]$$ fluxes

$$\mathbf{S} = \alpha\left[0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^{\,\mu}} - \Gamma^{\lambda}_{\mu\nu}g_{\lambda j}\right), \right.$$
$$\left. \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^{\,\mu}} - T^{\mu\nu}\Gamma^{\,0}_{\mu\nu}\right)\right]$$

curvature source
"gravitational
acceleration"
+ any interaction
terms etc.

GR Hydrodynamics

Primitive and Conserved Variables

$\rho, T, \epsilon, Y_e, v^i$ primitive

$\hat{D}, \hat{\tau}, \hat{D}Y_e, \hat{S}_i$ conserved

Why worry about primitive vars at all?
-> EOS is a function of the prim. vars!

Consequence:

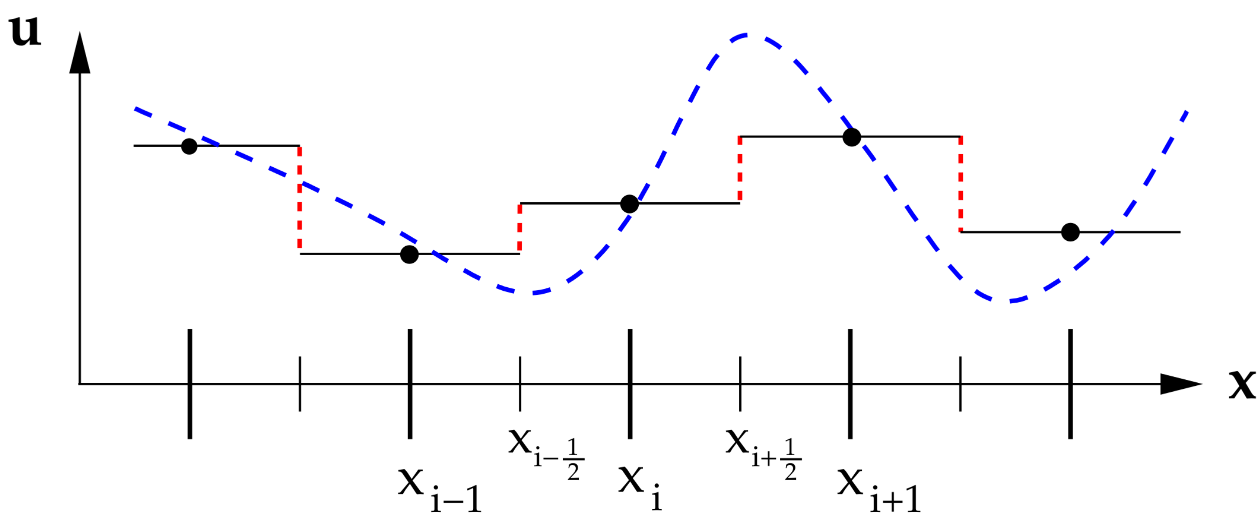
Must compute primitive variables from evolved conserved variables after each step!

There is no closed expression
-> must use Newton iteration to solve for primitive variables.

(This makes GR Hydro (and SR hydro) more expensive than Newtonian hydro)

GRHydro

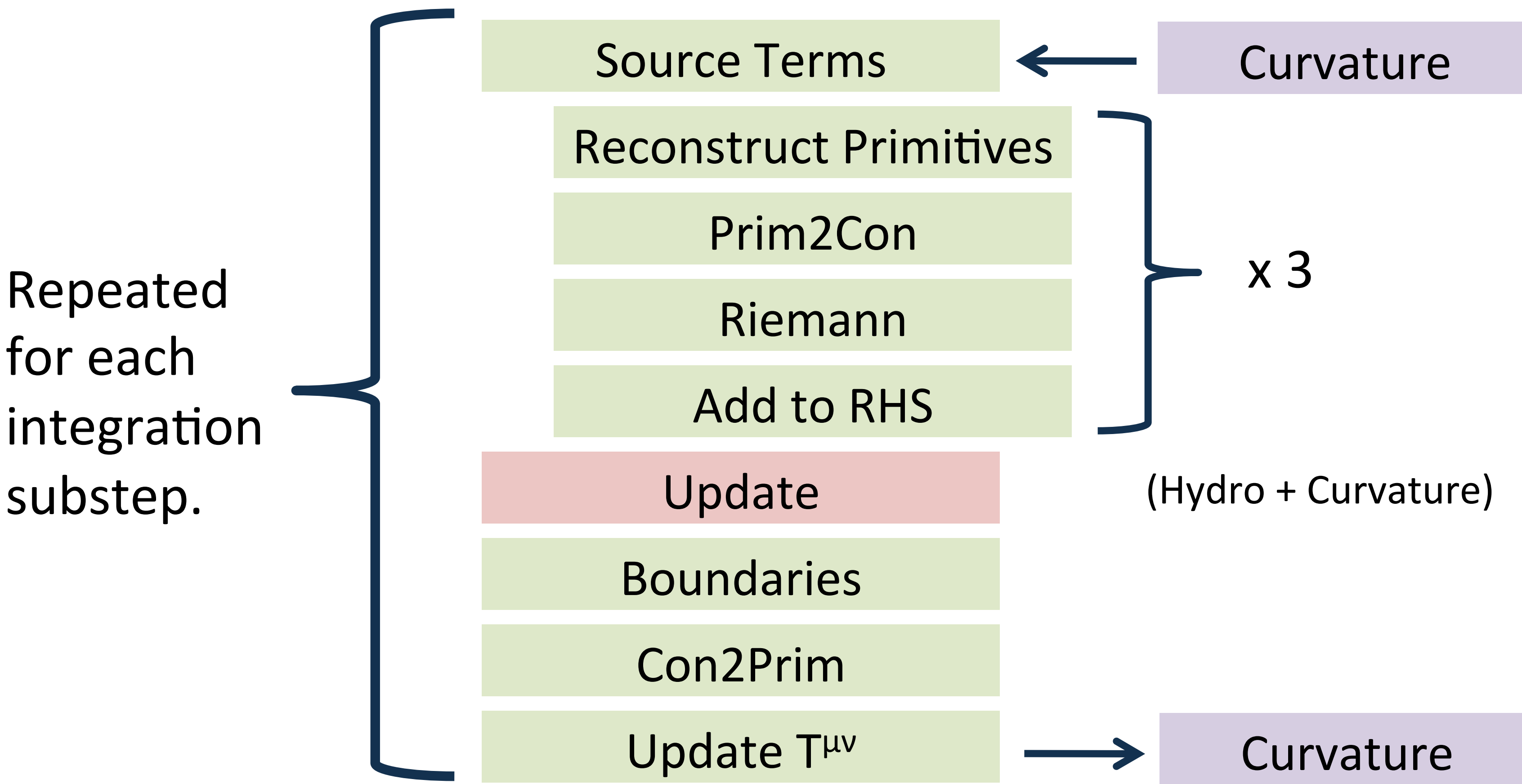
GRHydro, the EinsteinToolkit 3D GR Hydro Code



Basic scheme: high-resolution shock-capturing, finite-volume
data are cell averages, stored at cell centers
"reconstruction" (interpolation) to cell interfaces.
Approximate solution of **local** Riemann problems.

GRHydro

Simplified **GRHydro** Flow Chart



The Einstein Toolkit

http://einsteintoolkit.org

- Collection of open-source software components for the simulation and analysis of general-relativistic astrophysical systems.

Mösta+14
Löffler+12

If it can't be reproduced, it ain't science.

The Einstein Toolkit

http://einsteintoolkit.org

Mösta+14
Löffler+12

- Collection of open-source software components for the simulation and analysis of general-relativistic astrophysical systems.
- Supported by NSF via collaborative grant to Georgia Tech, LSU, RIT, and Caltech.
- ~110 users, 53 groups; ~10 active maintainers.
- Goals:
  - Reproducibility.
  - Build a community codebase for numerical relativity and computational relativistic astrophysics.
  - Enable new science by lowering technological hurdles for researchers with new ideas. Enable code verification/validation, physics benchmarking, regression testing.
  - Make it easy for users to take advantage of new technologies.
  - Provide cyberinfrastructure tools for code and data management.

The Einstein Toolkit

- Regular releases of stable code versions.
  Most recent: "Wheeler" release, May 2014
- Support via mailing list and weekly open conference calls.
- Working examples for BH mergers, NS mergers, isolated NSs, rotating, magnetized core collapse.

Mösta+14
Löffler+12

Available Components:

- Cactus (framework), Carpet (adaptive mesh refinement)
- GRHydro – GRMHD solver
- McLachlan – BSSN/Z4c spacetime solver
  (code auto-generated based on Mathematica script, GPU-enabled)
- Initial data solvers / importers
- Analysis tools (wave extraction, horizon finders, etc.)
- Visualization via VisIt (http://visit.llnl.gov)

Core ET Computational Science Tools

- GetComponents:
  - Collecting software components from distributed locations and version control repositories.
- SimFactory:
  - Automatic configuration and build of Einstein Toolkit on diverse machines.
  - Automatic simulation management.
- Formaline: Code provenance.
  - Snapshot of full source code and system configuration information stored in executable and/or git repository.

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Löffler+12

Workshop this Afternoon:

- Run a NSNS mergers simulation with the ET on Gordon!
- Solve the Tolman-Oppenheimer-Volkhoff equation to compute neutron star structure!
- Use the open-source GR1D code to study spherically-symmetric core collapse to neutron stars and black holes.

Numrel & GR Hydro Literature

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- Löffler+2012, The Einstein Toolkit: A Community Computational Infrastructure for Relativistic Astrophysics.
  Class. Quantum Grav., 29, 115001.
- Mösta+2014, GRHydro: a New Open-Source General-Relativistic Magnetohydrodynamics Code for the Einstein Toolkit, Class. Quantum Grav., 31, 015005.

Join the

American Physical Society

Topical Group in Gravitation (GGR)!!!

http://www.aps.org/units/ggr/

Now is a great time to join to help us gain
APS division status by 2015,
the centennial of General Relativity!

Supplemental Slides

CCSNe and Neutron Star Mergers:

Physics Ingredients:

Fully coupled!

Magneto-Hydrodynamics	Dynamics of the stellar gas.
General Relativity (GR)	**Gravity**
Nuclear and Neutrino Physics	**Nuclear EOS**, nuclear reactions & ν interactions.
Boltzmann Transport Theory	Neutrino transport.

-> Same *multi-physics* needs in CCSN and NS merger simulations!

-> **NS mergers: More relativistic situation -> GR is more important.**

Additional complication:

Multi-dimensional (3D) and multi-scale nature of CCSNe and mergers.

Simple Picture of Electron Capture

Simplest case: Capture on free protons, neutrinos escape

$$e^- + p \stackrel{(W)}{\rightarrow} \nu_e + n \qquad \mu_{\nu_e} = 0$$

capture if $\mu_e > \mu_n - \mu_p$

At zero T, non-degenerate nucleons: $\mu_e > 939.565\,\mathrm{MeV} - 938.272\,\mathrm{MeV} = 1.293\,\mathrm{MeV}$

In core collapse: Capture typically at μ_e ~ >10 MeV -> excess energy given to ν.

Capture rates: (see, e.g., Bethe et al. 1979, Bethe 1990, Burrows, Reddy & Thompson 2006)

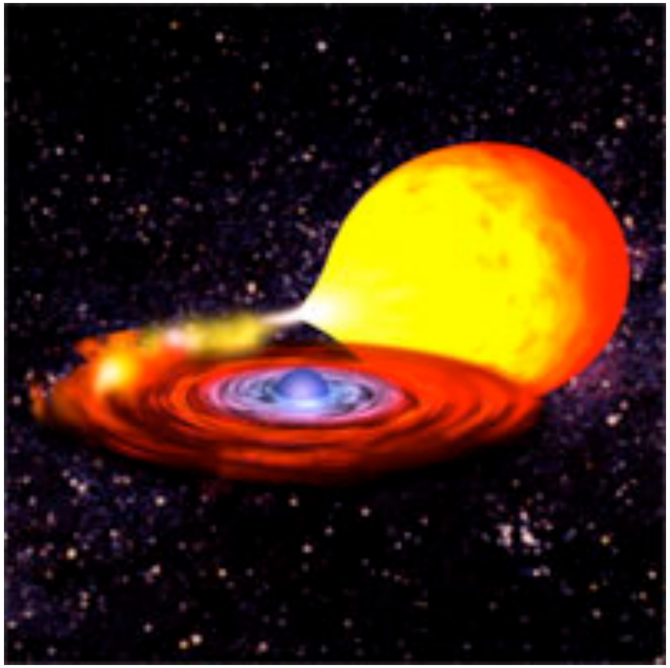
$$\frac{\partial}{\partial t} Y_e \propto \mu_e^5 \propto \rho^{5/3}$$

Complications:

- Capture on nuclei more complicated; can be blocked due to neutron shells filling up.
- Pauli blocking of low-energy states, since neutrinos don’t exactly leave immediately.

Type I X-Ray Bursts

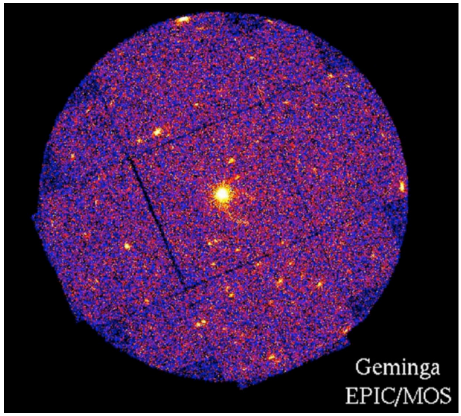
NASA



(see Lattimer 12 for review)

- Unstable He emission on NS surface.
- Rapidly rising X-ray burst (~1s), slow decay (~100s).
- Photosphere expansion:
  Radiation pressure pushes NS atmosphere (=photosphere), balances gravity.
- Observation + atmosphere models + distance
  -> radius and mass (but model dependent)

Quiescent NSs



XMM/Newton

- (Almost) Black-body UV/X-ray emission of young neutron stars.
- Depends on NS atmosphere composition, magnetic field, galactic UV/X-ray absorption. Need to know distance.
- Fits based on atmosphere models give radius and mass estimates.