Bohmian mechanics and cosmology

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Outline

I. Introduction to Bohmian mechanics

II. Bohmian mechanics and quantum gravity

III. Semi-classical approximation to quantum gravity based on Bohmian mechanics

IV. Quantum-to-classical transition in inflation theory

I. BOHMIAN MECHANICS

(a.k.a. pilot-wave theory, de Broglie-Bohm theory, . . .)

- De Broglie (1927), Bohm (1952)

- Particles moving under influence of the wave function.

- Dynamics:

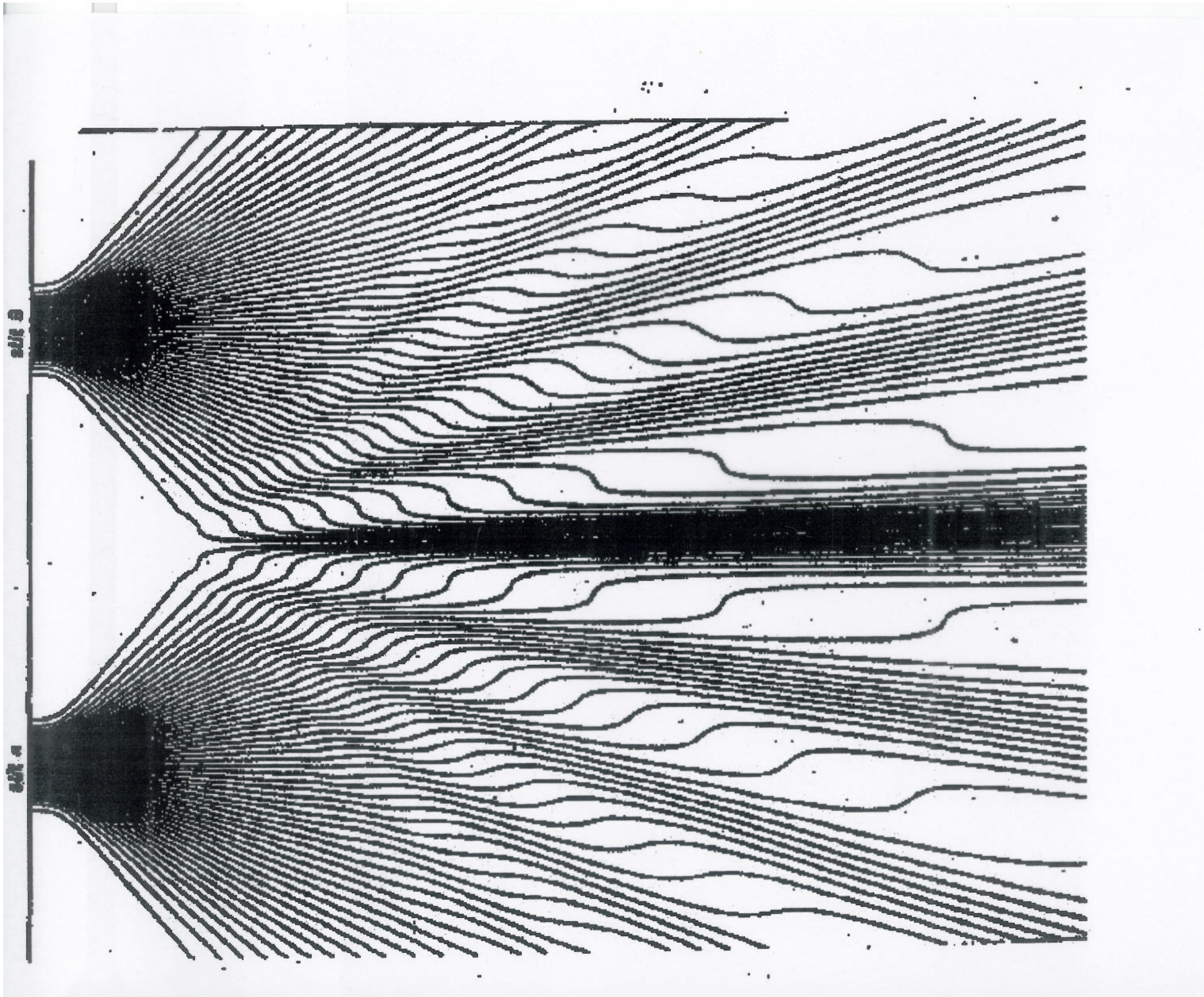
$$\mathrm{i}\hbar\partial_t\psi_t(x) = \left(-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\nabla_k^2 + V(x)\right)\psi_t(x)\,, \qquad x = (\mathbf{x}_1,\ldots,\mathbf{x}_N)$$

$$\frac{d\mathbf{X}_k(t)}{dt} = \mathbf{v}_k^{\psi_t}(X_1(t),\ldots,X_N(t))$$

where

$$\mathbf{v}_k^{\psi} = \frac{\hbar}{m_k}\mathrm{Im}\frac{\boldsymbol{\nabla}_k\psi}{\psi} = \frac{1}{m_k}\boldsymbol{\nabla}_k S, \qquad \psi = |\psi|e^{\mathrm{i}S/\hbar}$$

• Double Slit experiment:



- **Quantum equilibrium**:

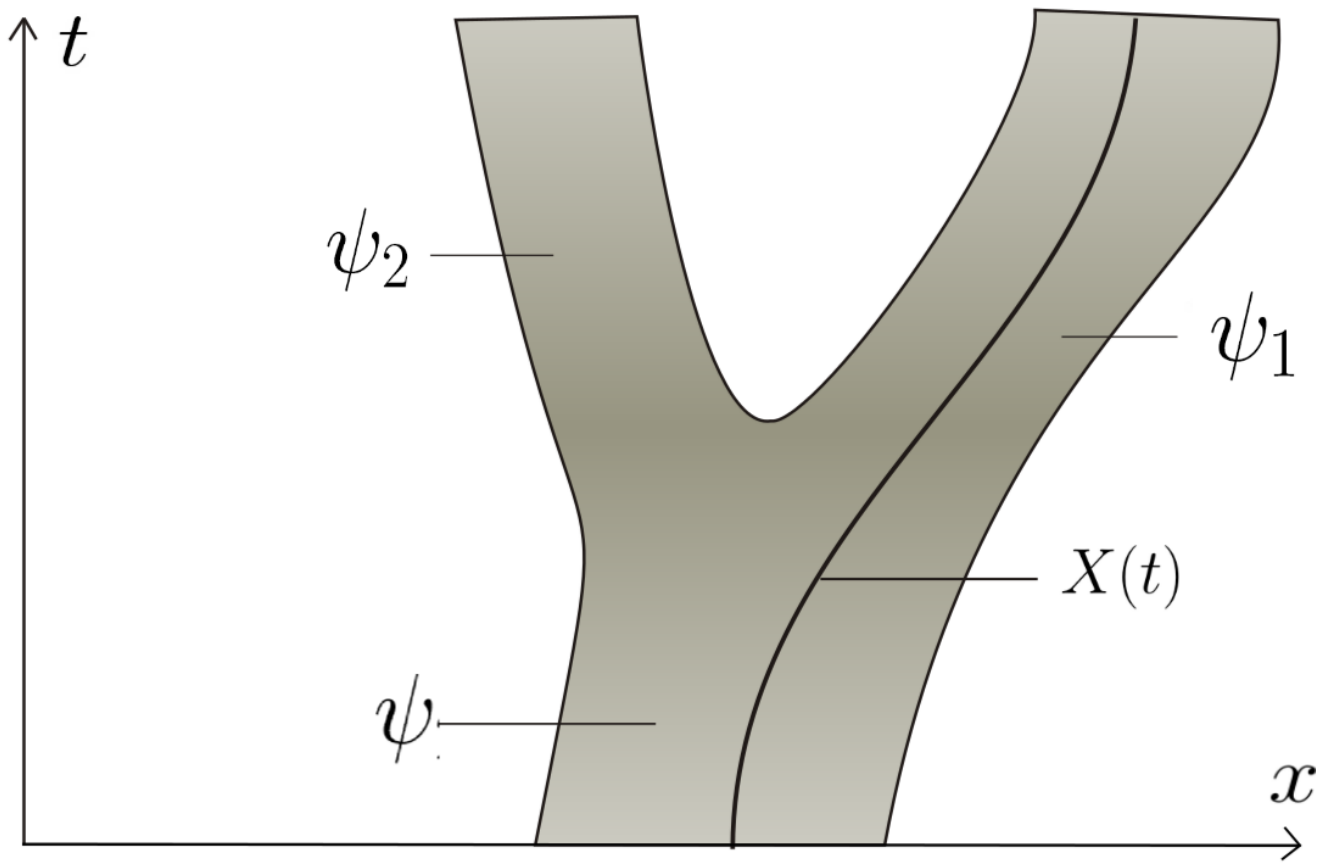
  - for an ensemble of systems with wave function ψ
  - distribution of particle positions $\rho(x) = |\psi(x)|^2$

Quantum equilibrium is preserved by the particle motion ($=$ equivariance), i.e.

$$\rho(x,t_0) = |\psi(x,t_0)|^2 \qquad \Rightarrow \qquad \rho(x,t) = |\psi(x,t)|^2 \qquad \forall t$$

Agreement with quantum theory in quantum equilibrium.

- **Effective collapse of the wave function**
  - Branching of the wave function: $\psi \rightarrow \psi_1 + \psi_2 \qquad \psi_1\psi_2 = 0$
  - Effective collapse $\psi \rightarrow \psi_1$ (ψ_2 does no longer effect the motion of the configuration X)



- **Wave function of subsystem: conditional wave function**

  Consider composite system: $\psi(x_1, x_2, t)$, $(X_1(t), X_2(t))$

  Conditional wave function for system 1:

$$\chi(x_1, t) = \psi(x_1, X_2(t), t)$$

  The trajectory $X_1(t)$ satisfies

$$\frac{dX_1(t)}{dt} = v^{\chi}(X_1(t), t)$$

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  Collapse of the conditional wave function

  Consider measurement:

  – Wave function system: $\psi(x) = \sum_i c_i \psi_i$

  (ψ_i are the eigenstates of the operator that is measured)

  – Wave function measurement device: $\phi(y)$

  – During measurement:

  Total wave function: $\psi(x)\phi(y) \to \sum_i c_i \psi_i(x)\phi_i(y)$

  Conditional wave function: $\psi(x) \to \psi_i(x)$

- **Classical limit:**

$$\dot{\mathbf{x}} = \frac{1}{m}\boldsymbol{\nabla} S \qquad \Rightarrow \qquad m\ddot{\mathbf{x}} = -\boldsymbol{\nabla}(V + Q)$$

$$\psi = |\psi| e^{\mathrm{i}S/\hbar}, \qquad Q = -\frac{\hbar^2}{2m}\frac{\nabla^2|\psi|}{|\psi|} = \text{quantum potential}$$

Classical trajectories when $|\boldsymbol{\nabla} Q| \ll |\boldsymbol{\nabla} V|$.

- **Non-locality**:

$$\frac{d\mathbf{X}_k(t)}{dt} = \mathbf{v}_k^{\psi_t}(X_1(t), \ldots, X_N(t))$$

$\rightarrow$ Velocity of one particle at a time t depends on the positions of all the other particles at that time, no matter how far they are.

Illustration of non-locality (Rice, AJP 1996)

Consider first a single particle

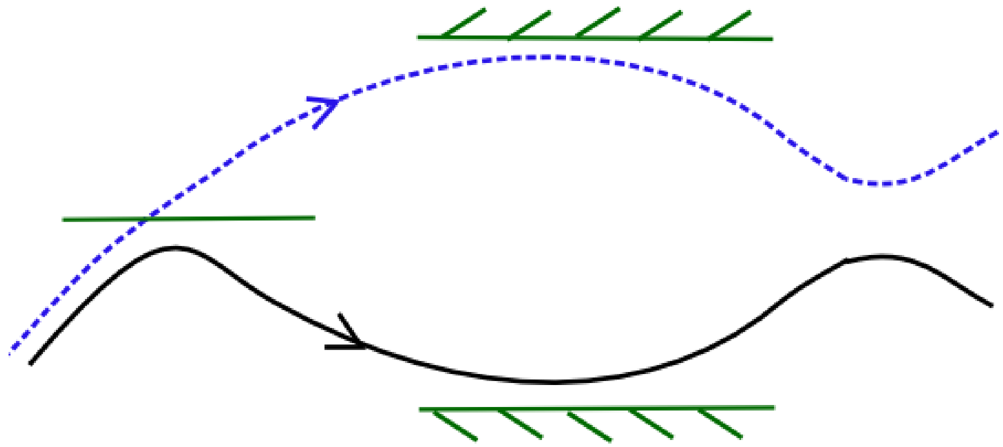
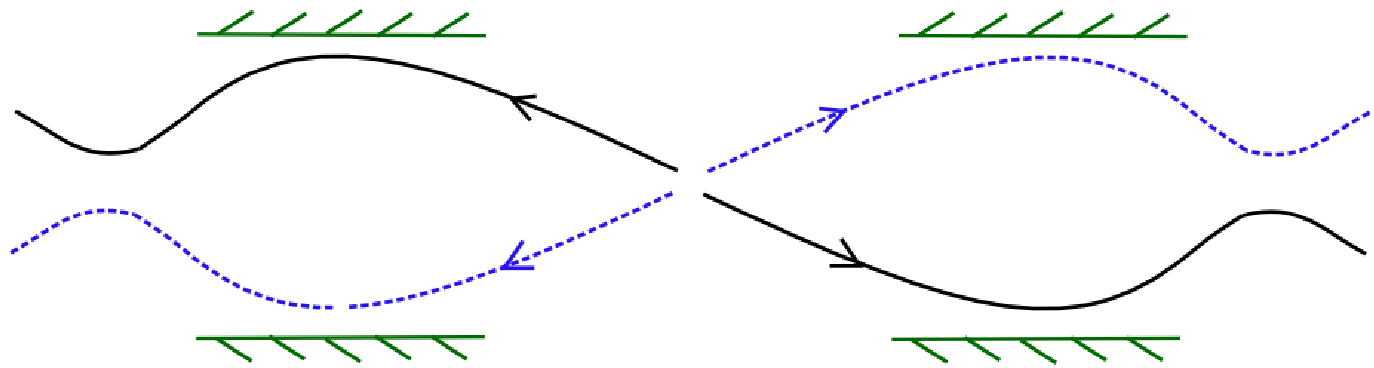


Illustration of non-locality (Rice, AJP 1996)

Consider the entangled state $|\nwarrow\rangle|\searrow\rangle + |\swarrow\rangle|\nearrow\rangle$



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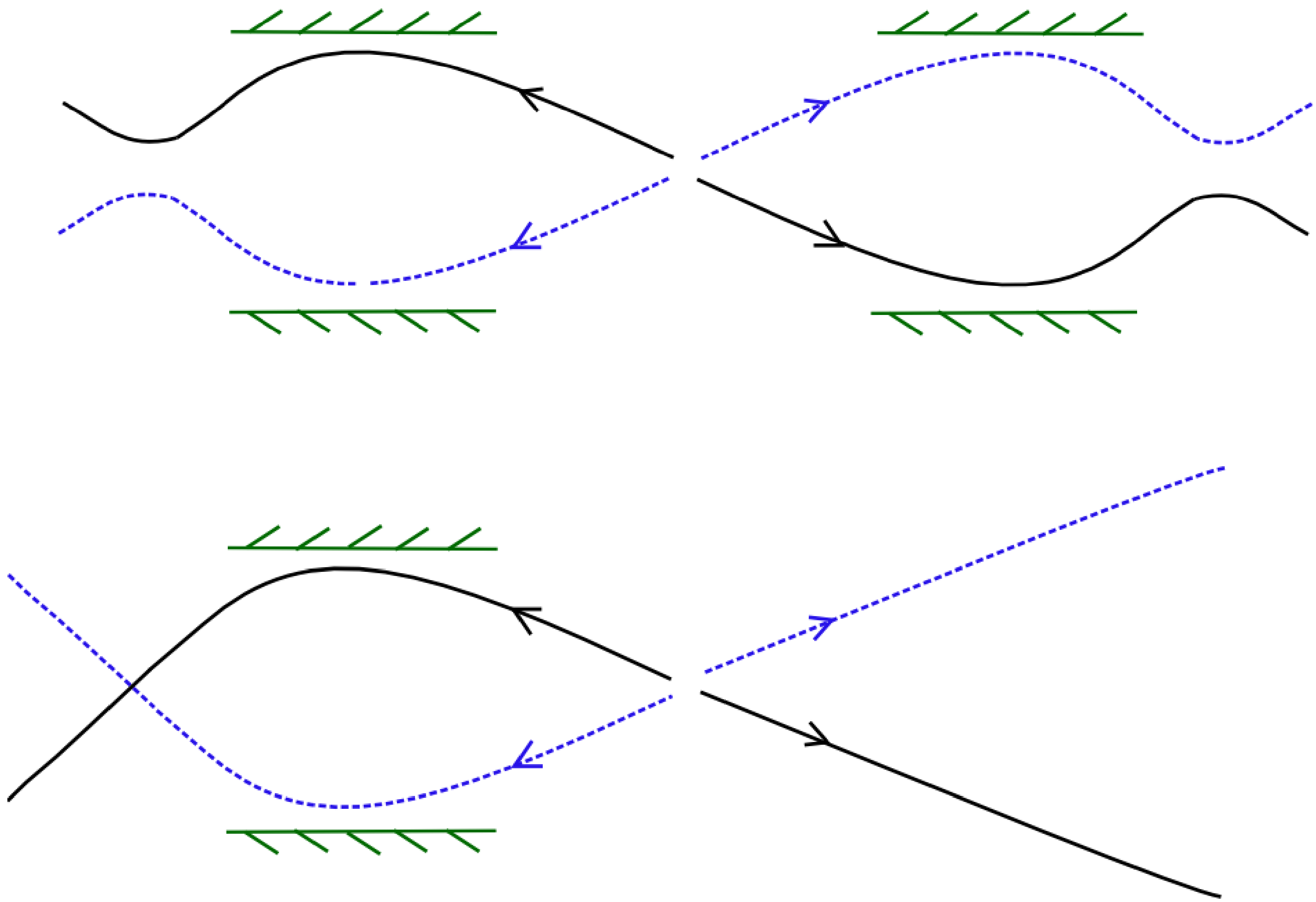

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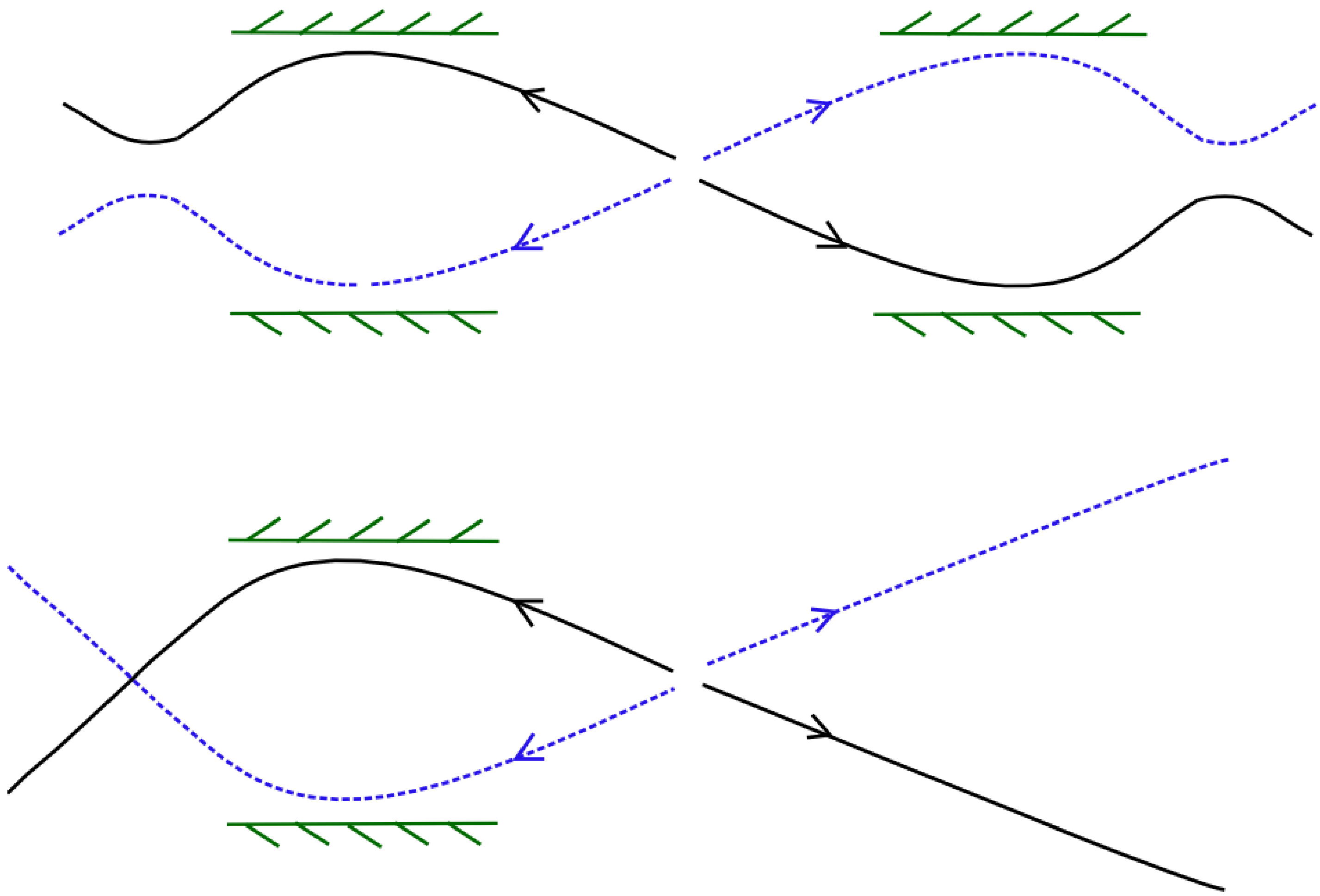

Non-local, but no faster than light signalling!

- **Extensions to quantum field theory**
  - Two natural possible ontologies: particles and fields. Particles seem to work better for fermions, fields for bosons.
  - Example: scalar field

    Hamiltonian:

$$\widehat{H} = \frac{1}{2}\int d^3x \left(\widehat{\Pi}^2 + (\nabla\widehat{\phi})^2 + m^2\widehat{\phi}^2\right)\,, \qquad [\widehat{\phi}(\mathbf{x}), \widehat{\Pi}(\mathbf{y})] = \mathrm{i}\delta(\mathbf{x}-\mathbf{y})$$

    Functional Schrödinger representation:

$$\widehat{\phi}(\mathbf{x}) \to \phi(\mathbf{x})\,, \qquad \widehat{\pi}(\mathbf{x}) \to -\mathrm{i}\frac{\delta}{\delta\phi(\mathbf{x})}$$

$$\mathrm{i}\frac{\partial\Psi(\phi,t)}{\partial t} = \frac{1}{2}\int d^3x \left(-\frac{\delta^2}{\delta\phi^2} + (\nabla\phi)^2 + m^2\phi^2\right)\Psi(\phi,t)\,.$$

    Bohmian field $\phi(\mathbf{x})$ with guidance equation:

$$\frac{\partial\phi(\mathbf{x},t)}{\partial t} = \frac{\delta S(\phi,t)}{\delta\phi(\mathbf{x})}\bigg|_{\phi=\phi(\mathbf{x},t)}, \qquad \Psi = |\Psi|e^{\mathrm{i}S}$$

    Similarly for other bosonic fields (see Struyve (2010) for a review):

    electromagnetic field: $\Psi(A)$, $A(\mathbf{x})$, gravity: $\Psi(g)$, $g(\mathbf{x})$, $\ldots$

II. QUANTUM GRAVITY

Canonical quantization of Einstein's theory for gravity:

$$g^{(3)}(x) \to \widehat{g}^{(3)}(x)$$

In funcional Schrödinger picture:

$$\Psi = \Psi(g^{(3)})$$

Satisfies the Wheeler-De Witt equation and constraints:

$$\mathrm{i}\frac{\partial \Psi}{\partial t} = \widehat{H}\Psi = 0$$

$$\widehat{H}_i \Psi = 0$$

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Conceptual problems:

1. Problem of time: There is no time evolution, the wave function is static.

   (How can we tell the universe is expanding or contracting?)

2. Measurement problem: We are considering the whole universe. There are no outside observers or measurement devices.

3. What is the meaning of space-time diffeomorphism invariance? (The constraints $\widehat{H}_i \Psi = 0$ only express invariance under spatial diffeomorphisms.)

Bohmain approach

In a Bohmian approach we have an actual 3-metric $g^{(3)}$ which satisfies:

$$\dot{g}^{(3)} = v^{\Psi}(g^{(3)})$$

This solves problems 1:

- We can tell whether the universe is expanding or not, whether it goes into a singularity or not, etc.
- We can derive time dependent Schrödinger equation for conditional wave function. E.g. suppose gravity and scalar field. Conditional wave functional for scalar field

$$\Psi_s(\phi, t) = \Psi(\phi, g^{(3)}(t))$$

is time-dependent if $g^{(3)}(t)$ is time-dependent.

It also solves problem 2. Does it solve problem 3?

For more details, see: Goldstein & Teufel, Callender & Weingard, Pinto-Neto, ...

III. SEMI-CLASSICAL GRAVITY

Apart from the conceptual difficulties with the quantum treatment of gravity, there are also technical problems: finding solutions to Wheeler-DeWitt equation, doing perturbation theory, etc. Therefore one often resorts to semi-classical approximations:

→ **Matter is treated quantum mechanically**, as quantum field on curved space-time.

E.g. scalar field:

$$\mathrm{i}\partial_t \Psi(\phi, t) = \widehat{H}(\phi, g)\Psi(\phi, t)$$

→ **Gratvity is treated classically**, described by

$$G_{\mu\nu}(g) = \frac{8\pi G}{c^4}\langle\Psi|\widehat{T}_{\mu\nu}(\phi, g)|\Psi\rangle$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$$

Is there a better semi-classical approximation based on Bohmian mechanics?

In Bohmian mechanics matter is described by $\Psi(\phi)$ and actual scalar field $\phi_B(\mathbf{x}, t)$.

Proposal for semi-classical theory:

$$G_{\mu\nu}(g) = \frac{8\pi G}{c^4} T_{\mu\nu}(\phi_B, g)$$

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(In non-relativistic Bohmian mechanics energy is not conserved.)

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Similar situation in scalar electrodynamics:

Quantum matter field described by $\Psi(\phi)$ and actual scalar field $\phi_B(\mathbf{x}, t)$. Semi-classical theory:

$$\partial_\mu F^{\mu\nu} = j^\nu(\phi_B)$$

$\rightarrow$ In general doesn't work because $\partial_\nu j^\nu(\phi_B) \neq 0$!

Semi-classical approximation to non-relativistic quantum mechanics

- System 1: quantum mechanical. System 2: classical

Usual approach (mean field):

$$\mathrm{i}\partial_t \psi(x_1,t) = \left(-\frac{\nabla_1^2}{2m_1} + V(x_1, X_2(t))\right)\psi(x_1,t)$$

$$m_2 \ddot{X}_2(t) = \langle\psi|F_2(x_1, X_2(t))|\psi\rangle = \int dx_1 |\psi(x_1,t)|^2 F_2(x_1, X_2(t))\,, \quad F_2 = -\nabla_2 V$$

$\rightarrow$ **backreaction through mean force**

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Bohmian approach:

$$\mathrm{i}\partial_t\psi(x_1,t) = \left(-\frac{\nabla_1^2}{2m_1} + V(x_1, X_2(t))\right)\psi(x_1,t)$$

$$\dot{X}_1(t) = v_1^{\psi}(X_1(t),t)\,, \qquad m_2\ddot{X}_2(t) = F_2(X_1(t), X_2(t))$$

$\rightarrow$ **backreaction through Bohmian particle**

- Prezhdo and Brookby (2001):

  Bohmian approach yields better results than usual approach:

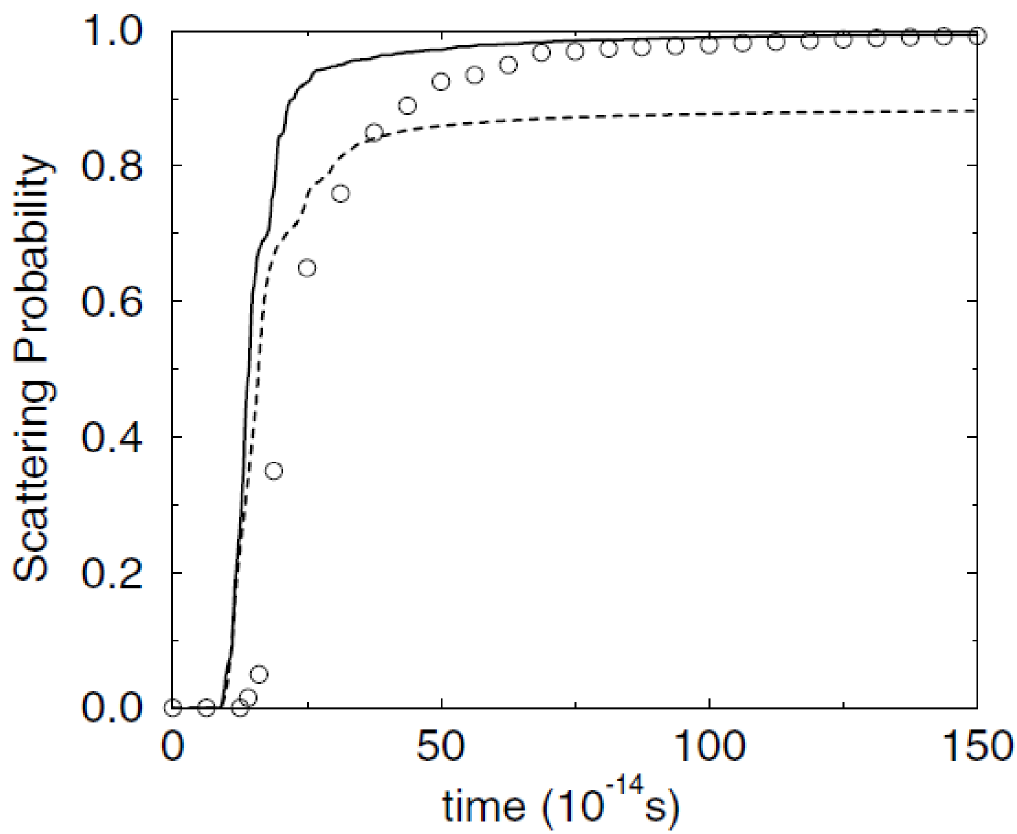


FIG. 1. The time-dependent scattering probability P_s, Eq. (16), for the model problem detailed in the text obtained for the incident energy of 20 kJ/mol using exact quantum dynamics (circles), mean-field dynamics (dashed curve), and the Bohmian quantum-classical technique (solid curve).

- **Derivation of Bohmian semi-classical approximation**

Full quantum mechanical description:

$$\mathrm{i}\partial_t \psi(x_1, x_2, t) = \left(-\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} + V(x_1, x_2)\right)\psi(x_1, x_2, t)$$

$$\dot{X}_1(t) = v_1^{\psi}(X_1(t), X_2(t), t)\,, \quad \dot{X}_2(t) = v_2^{\psi}(X_1(t), X_2(t), t)$$

Conditional wave function $\chi(x_1, t) = \psi(x_1, X_2(t), t)$ satisfies

$$\mathrm{i}\partial_t \chi(x_1, t) = \left(-\frac{\nabla_1^2}{2m_1} + V(x_1, X_2(t))\right)\chi(x_1, t) + I(x_1, t)$$

and particle two:

$$m_2 \ddot{X}_2(t) = -\nabla_2 V(X_1(t), x_2)\Big|_{x_2 = X_2(t)} - \nabla_2 Q(X_1(t), x_2)\Big|_{x_2 = X_2(t)}$$

$\rightarrow$ Semi-classical approximation follows when I and $-\nabla_2 Q$ are negligible

(e.g. when particle 2 is much heavier than particle 1)

Bohmian semi-classical approximation to scalar quantum electrodynamics

- Schrödinger equation for matter:

$$\mathrm{i}\partial_t \Psi(\phi, t) = \widehat{H}(\phi, A)\Psi(\phi, t)$$

Guidance equation

$$\dot{\phi} = v^{\Psi}(\phi, t)$$

Classical Maxwell equations for with quantum correction:

$$\partial_\mu F^{\mu\nu} = j^\nu + j^\nu_Q\,,$$

Is consistent since: $\partial_\mu(j^\mu + j^\mu_Q) = 0$**.**

$\rightarrow$ **Crucial in the derivation was that gauge was eliminated!**
How to eliminate it in canonical quantum gravity?
(in this case: gauge = spatial diffeomorphism invariance).

Bohmian semi-classical approximation to mini-superspace model

- Restriction to homogeneous and isotropic (FLRW) metrics and fields:
  - Gravity: $\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2\mathrm{d}\Omega_3^2$
  - Matter: $\phi = \phi(t)$

  Wheeler-DeWitt equation:

$$(H_G + H_M)\psi = 0\,,$$

$$H_G = \frac{1}{4a^2}\partial_a(a\partial_a) + a^3 V_G\,, \qquad H_M = -\frac{1}{2a^3}\partial_\phi^2 + a^3 V_M$$

  Guidance equations:

$$\dot{a} = -\frac{1}{2a}\partial_a S\,, \qquad \dot{\phi} = \frac{1}{a^3}\partial_\phi S$$

- Semi-classical approximation:

$$\mathrm{i}\partial_t\psi = H_M\psi\,, \qquad \dot{\phi} = \frac{1}{a^3}\partial_\phi S$$

  and Friedmann equation with quantum correction:

$$\frac{\dot{a}^2}{a^2} = \frac{\dot{\phi}^2}{2} + V_M + V_G + Q$$

IV. QUANTUM-TO-CLASSICAL TRANSITION IN INFLATION THEORY

Cosmological perturbations

- Inflaton field: $\varphi(\mathbf{x},\eta) = \varphi_0(\eta) + \delta\varphi(\mathbf{x},\eta)$

  Metric with scalar perturbations, in the longitudinal gauge:

$$\mathrm{d}s^2 = a^2(\eta)\left\{[1+2\phi(\eta,\mathbf{x})]\,\mathrm{d}\eta^2 - [1-2\phi(\eta,\mathbf{x})]\,\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right\},$$

- Gauge invariant Mukhanov-Sasaki variable which describes perturbations:

$$y \equiv a\left[\delta\varphi + \frac{\varphi_0'}{\mathcal{H}}\phi\right],$$

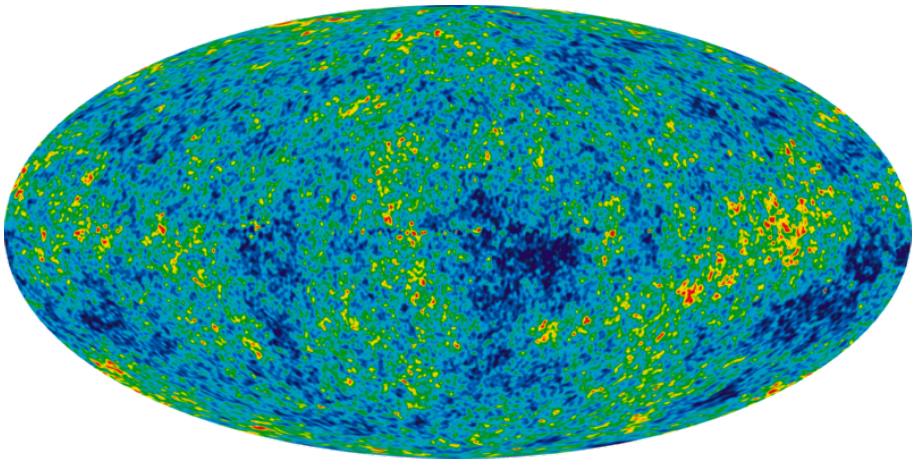
  with $\mathcal{H} = \frac{a'}{a}$ the comoving Hubble parameter. Its classical equation of motion is:

$$y'' - \nabla^2 y - \frac{z''}{z}y = 0 \qquad (z = a\varphi_0'/\mathcal{H})$$

- So we have 3 variables: a, φ_0 and y.
  - a and φ_0 are treated classically and independent of y
  - y is quantized. The assumed quantum state $\Psi(y)$ is the Bunch-Davies vacuum.

The quantum vacuum fluctuations give rise to

- the fluctuations in CMB
- to structures such as galaxies, clusters of galaxies, etc.



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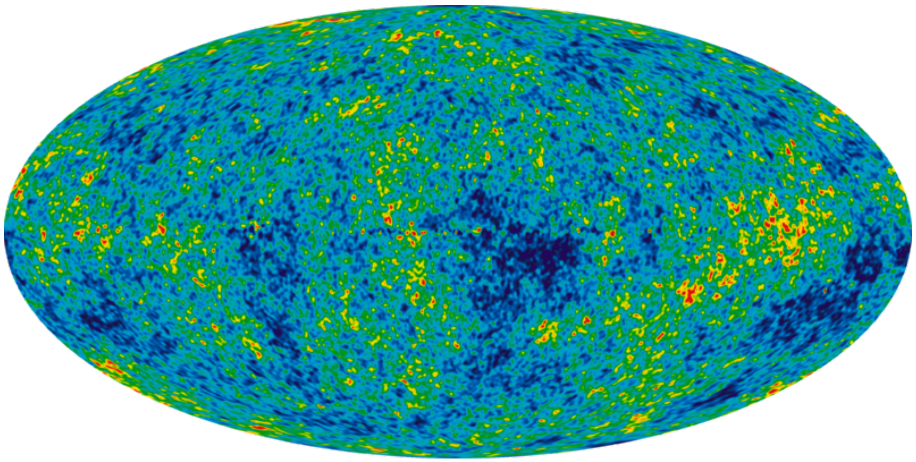


However:

$\rightarrow$ How does the vacuum state of the perturbations, which is homogeneous and isotropic, gives rise to perturbations which are inhomogeneous and anisotropic?

$\rightarrow$ How do the quantum fluctuations become classical fluctuations?

According to standard quantum theory this can only be achieved by collapse of the wave function. But collapse is supposed to happen upon measurement. But when exactly does a measurement happen? Which processes count as measurements in the early universe?

$\longrightarrow$ Measurement problem!

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Possible solutions:

collapse theories (Sudarsky), many worlds, Bohmian mechanics

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$\rightarrow$ Measurement problem

$\rightarrow$ Is especially severe in cosmological context! Which processes count as measurement in the early universe?

Possible solutions:

collapse theories (Sudarsky), many worlds, Bohmian mechanics

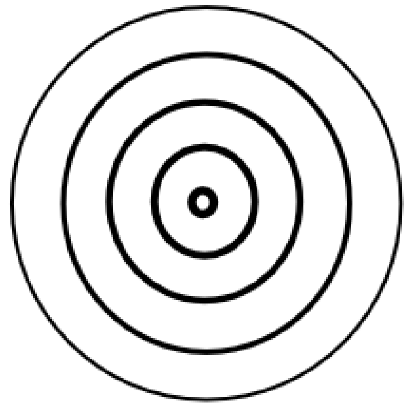
$\rightarrow$ We illustrate the problem and possible solutions in the simple cases of

- a decaying atom
- the inverted harmonic oscillator

(For Bohmian treatment of the problem in inflation theory, see Pinto-Neto, Santos, Struyve 2012)

Decaying atom

Consider a decaying atom which emits a photon described by a spherically symmetric wave function:



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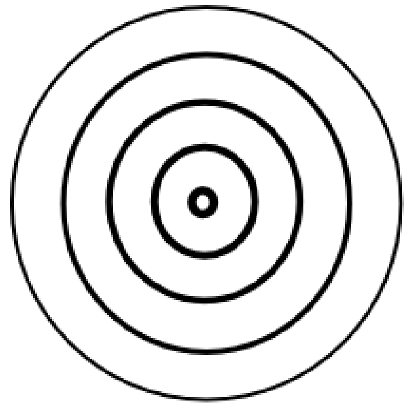
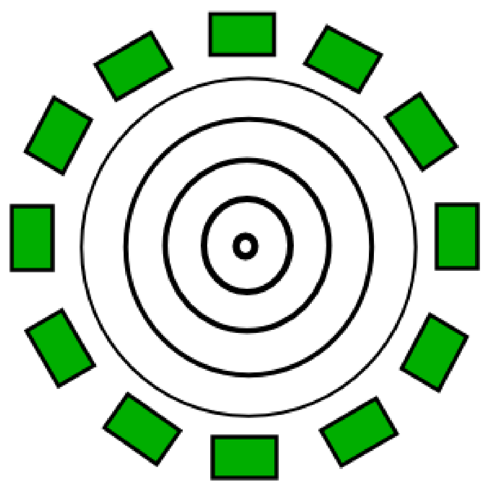


With detectors:



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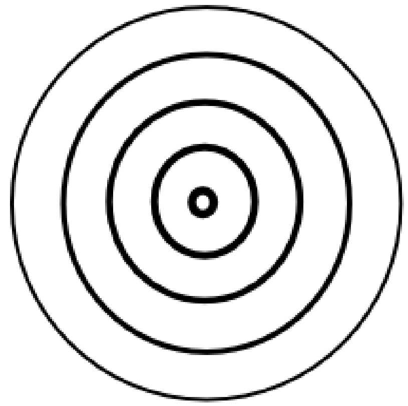

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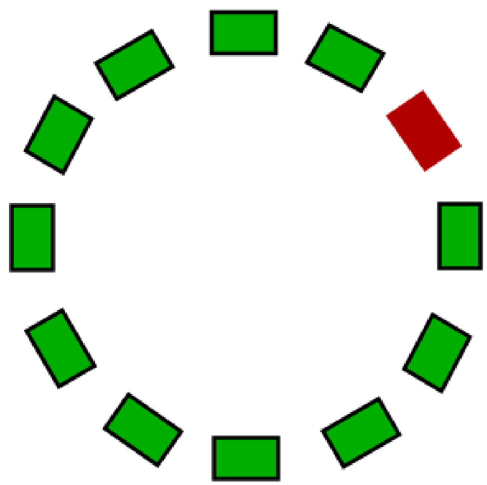
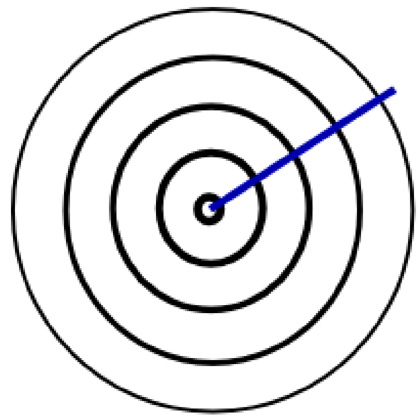


$\rightarrow$ according to standard quantum theory collapse breaks the symmetry

Bohmian description:

Without detectors:



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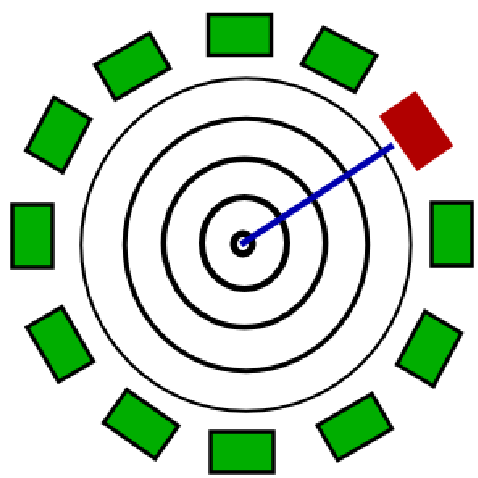
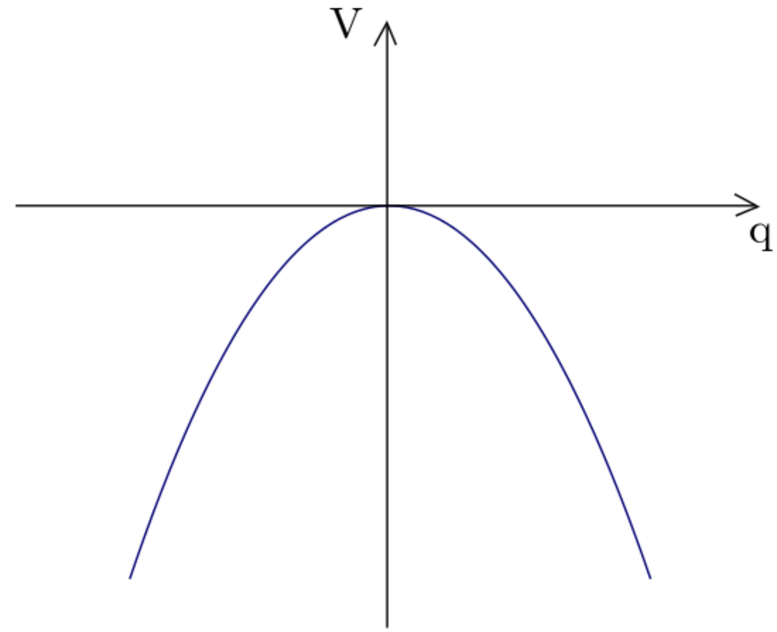


$\rightarrow$ actual particle breaks the symmetry

Inverted harmonic oscillator (e.g. Albrecht et al. 1994)

Classical treatment



- Potential: $V = -\frac{q^2}{2}$
- Equation of motion: $\ddot{q} = q$
- Possible trajectories: $q = A\mathrm{e}^{t} + B\mathrm{e}^{-t}$

Inverted harmonic oscillator (e.g. Albrecht et al. 1994)

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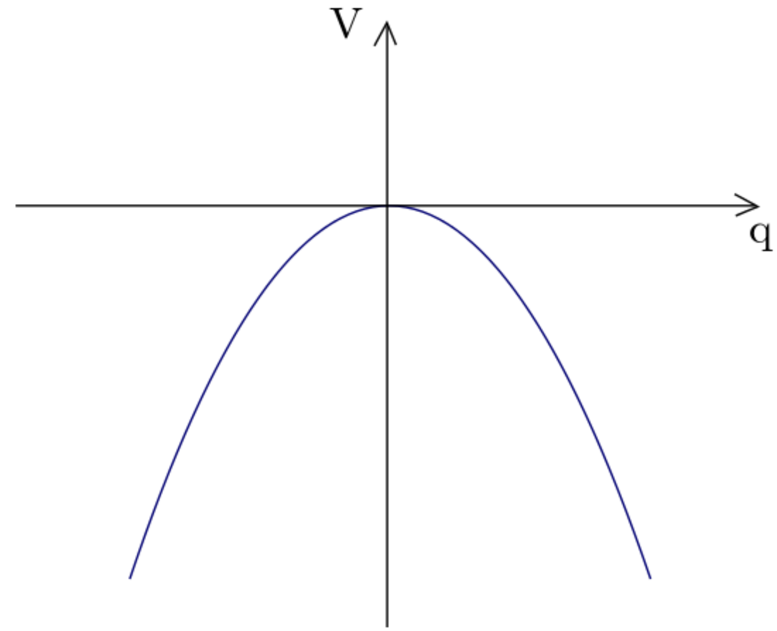

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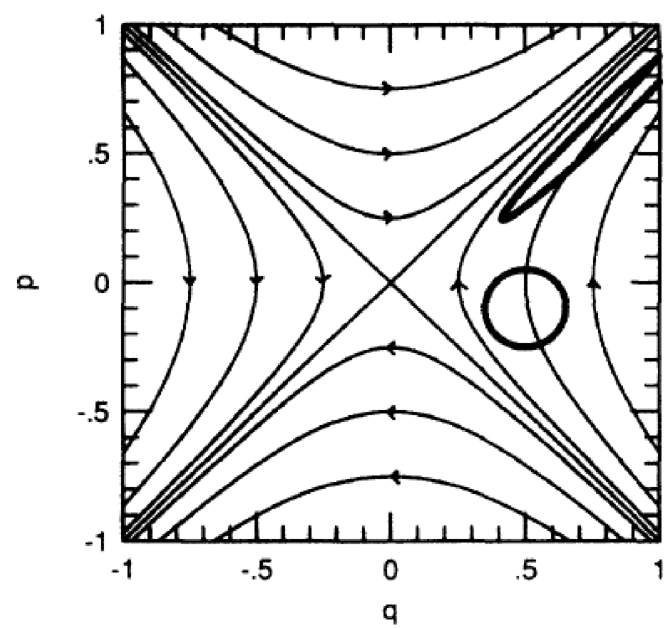
In phase space:

$$q = A\mathrm{e}^{t} + B\mathrm{e}^{-t}, \quad p = A\mathrm{e}^{t} - B\mathrm{e}^{-t}$$

$$q \approx p \approx A\mathrm{e}^{t} \qquad \text{for} \qquad t \gg 1$$

$$\Rightarrow \quad \textbf{squeezing}$$



Quantum mechanics

Squeezed state:

$$\psi(q,t) = N \exp\left(- \frac{(B - \mathrm{i}C)}{2} q^2 - \mathrm{i}\frac{B}{2} t \right)$$

$$N = \left(\frac{B}{\pi}\right)^{\frac{1}{4}}, \qquad B = \frac{1}{\cosh 2t}, \qquad C = \tanh 2t$$

Note

$$\Delta q^2 = \frac{1}{2B}, \qquad \Delta p^2 = \frac{B}{2} + \frac{C^2}{2B}$$

$$\text{For } \ t = 0 : \qquad \Delta q^2 = \Delta p^2 = \frac{1}{2}$$

$$\text{For } \ t \gg 1 : \qquad \Delta q^2 \approx \Delta p^2 \gg 1$$

$\rightarrow$ Initially minimum uncertainty in q and p. However, both spread in time!

$\rightarrow$ The wave function is not peaked around a classical trajectory!

How can it correspond to a classical system?

Common classicality arguments

1. Commuting observables

Heisenberg operators (time evolution $\widehat{O}(t) = \mathrm{e}^{\mathrm{i}\widehat{H}t}\widehat{O}(0)\mathrm{e}^{-\mathrm{i}\widehat{H}t}$):

$$\widehat{q}(t) = \widehat{A}\mathrm{e}^{t} + \widehat{B}\mathrm{e}^{-t}\,, \qquad \widehat{p}(t) = \widehat{A}\mathrm{e}^{t} - \widehat{B}\mathrm{e}^{-t}$$

(with $\widehat{A} = \frac{1}{2}\left(\widehat{q}(0) + \widehat{p}(0)\right)$, $\widehat{B} = \frac{1}{2}\left(\widehat{q}(0) - \widehat{p}(0)\right)$)

For $t \gg 1$:

$$\widehat{q}(t) \approx \widehat{p}(t) \approx \widehat{A}\mathrm{e}^{t}$$

Hence

$$[\widehat{q}(t), \widehat{p}(t)] \approx 0 \qquad \Rightarrow \qquad \textbf{Classicality}$$

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1. Commuting observables

Heisenberg operators (time evolution $\widehat{O}(t) = \mathrm{e}^{\mathrm{i}\widehat{H}t}\widehat{O}(0)\mathrm{e}^{-\mathrm{i}\widehat{H}t}$):

$$\widehat{q}(t) = \widehat{A}\mathrm{e}^{t} + \widehat{B}\mathrm{e}^{-t}\,, \qquad \widehat{p}(t) = \widehat{A}\mathrm{e}^{t} - \widehat{B}\mathrm{e}^{-t}$$

(with $\widehat{A} = \frac{1}{2}\left(\widehat{q}(0) + \widehat{p}(0)\right)$, $\widehat{B} = \frac{1}{2}\left(\widehat{q}(0) - \widehat{p}(0)\right)$)

For $t \gg 1$:

$$\widehat{q}(t) \approx \widehat{p}(t) \approx \widehat{A}\mathrm{e}^{t}$$

Hence

$$[\widehat{q}(t), \widehat{p}(t)] \approx 0 \qquad \Rightarrow \qquad \textbf{Classicality}$$

However

$$[\widehat{q}(t), \widehat{p}(t)] = \mathrm{i} \not\approx 0$$

Similarly: free particle

Heisenberg operators:

$$\widehat{x}(t) = \widehat{x}(0) + \frac{t}{m}\widehat{p}(0)\,, \qquad \widehat{p}(t) = \widehat{p}(0)$$

For large t/m:

$$\widehat{x}(t) \approx \frac{t}{m}\widehat{p}(0)$$

Hence

$$[\widehat{x}(t), \widehat{p}(t)] \approx 0 \qquad \Rightarrow \qquad \text{Classicality}$$

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However

$$[\widehat{x}(t), \widehat{p}(t)] = \mathrm{i} \not\approx 0$$

A correct argument:

$$\begin{aligned}\Delta x(t)^2 &= \Delta x(0)^2 + \frac{t}{m}\Big(\langle\{\widehat{x}(0), \widehat{p}(0)\}\rangle - \langle\widehat{x}(0)\rangle\langle\widehat{p}(0)\rangle\Big) + \frac{t^2}{m^2}\Delta p(0)^2 \\ &\approx \Delta x(0)^2 \qquad\qquad \text{for } \textit{small}\ \frac{t}{m}\end{aligned}$$

$\Rightarrow$ No spreading for a very massive particle for short enough times.

2. Wigner distribution:

$$\rho(q,p,t) = \frac{1}{\sqrt{\pi B}} |\psi(q,t)|^2 \exp\left(-\frac{(p-Cq)^2}{B} \right)$$
$$\to |\psi(q,t)|^2 \delta(p-q) \qquad \text{for} \qquad t \gg 1$$

$\to$ Is not peaked around one particular classical trajectory

$\to$ But:

- is positive (is usually not the case)
- satisfies Liouville equation $d\rho/dt = 0$ (is usually not the case)
- quantum mechanical expectation values equal classical averages over ρ

However, this does not mean classical limit is achieved!

3. WKB limit

With $\psi = |\psi| \mathrm{e}^{\mathrm{i}S}$:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2} + V + Q = 0\,,$$

$$V = -\frac{q^2}{2}\,, \qquad Q = \frac{B}{2}(1 - Bq^2)$$

For $t \gg 1$:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2} + V \approx 0\,,$$

$\rightarrow$ Formally same as classical Hamilton-Jacobi equation

But:

Does not imply we can assume a classical trajectory

4. Decoherence

Decoherence due to coupling with other degrees of freedom may yield decomposition of ψ into "classical wave packets". Collapse may select one of these.

Where does the decoherence come from in inflation theory?

- Interactions between sub and super Hubble modes (which would show up when treating the fluctuations up to second order).
- Interactions with the matter fields

De Broglie-Bohm description description of the inverted oscillator

$$\dot{q} = \nabla S \qquad \Rightarrow \qquad \ddot{q} = F_C + F_Q$$

Classical force: $F_C = q$

Quantum force: $F_Q = qB^2$

Ratio:

$$\frac{F_Q}{F_C} = B^2 \ \ \rightarrow 0 \quad \text{for} \quad t \gg 1 \qquad \rightarrow \textbf{classical behaviour}$$

More precisely:

$$\begin{aligned} q(t) &\sim \sqrt{\mathrm{e}^{2t} + \mathrm{e}^{-2t}} \\ &\sim \mathrm{e}^{t} \quad \text{for} \quad t \gg 1 \end{aligned}$$

$\rightarrow$ No appeal to decoherence!

$\rightarrow$ If there is decoherence of the expected type, then this will not affect the classicality.