The Inflationary Universe

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A Mystery

$$ds^2 = a(\eta)^2 \left[-\left(1+2\Psi\right) d\eta^2 + \left(1+2\Phi\right) \delta_{ij} dx^i dx^j \right]$$

$$P(k) = Ak^{n_s - 1}$$

Why?

A Mystery

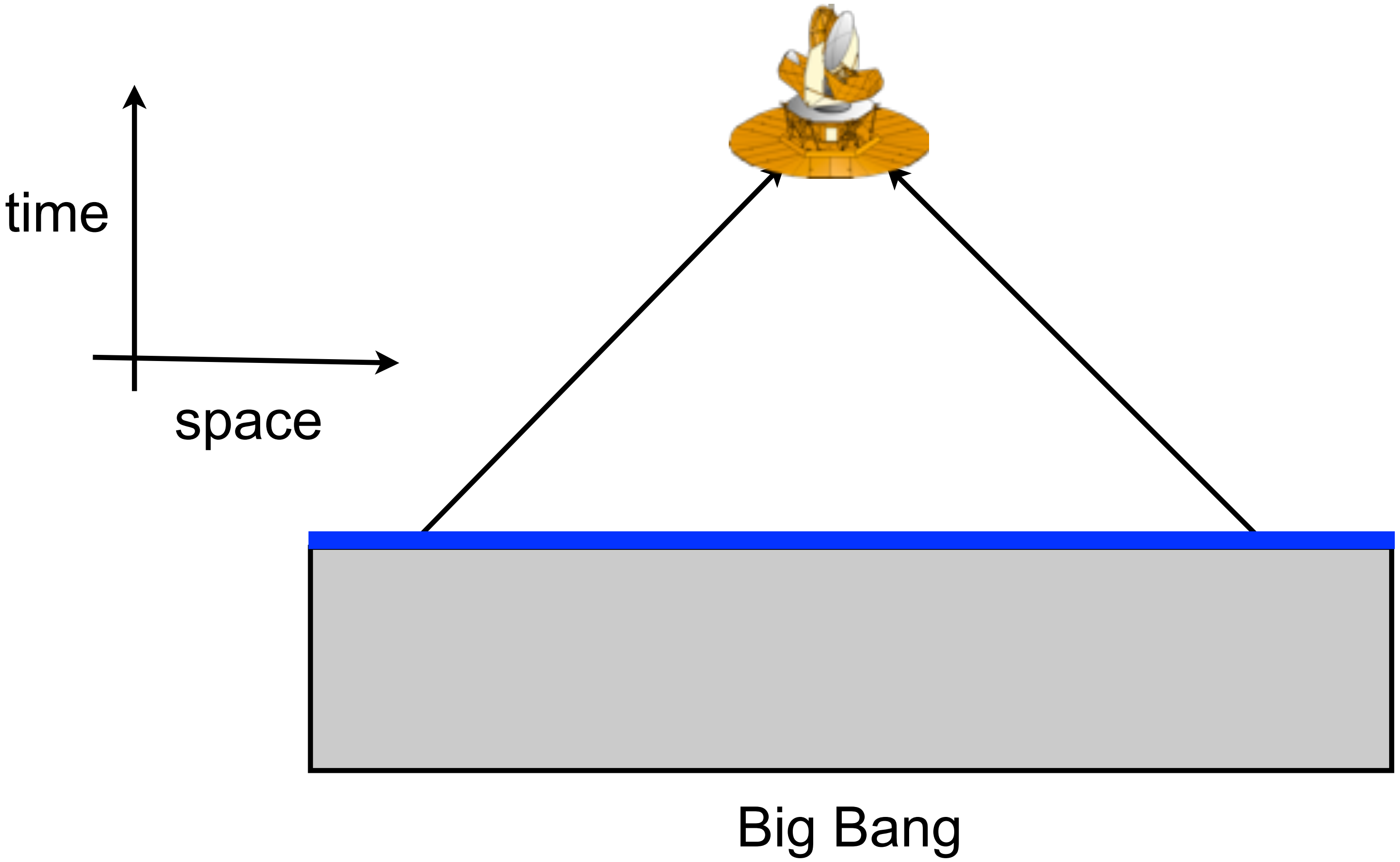
$$ds^2 = a(\eta)^2 \left[- \left(1 + 2\Psi\right) d\eta^2 + \left(1 + 2\Phi\right) \delta_{ij} dx^i dx^j \right]$$

$$P(k) = Ak^{n_s - 1}$$

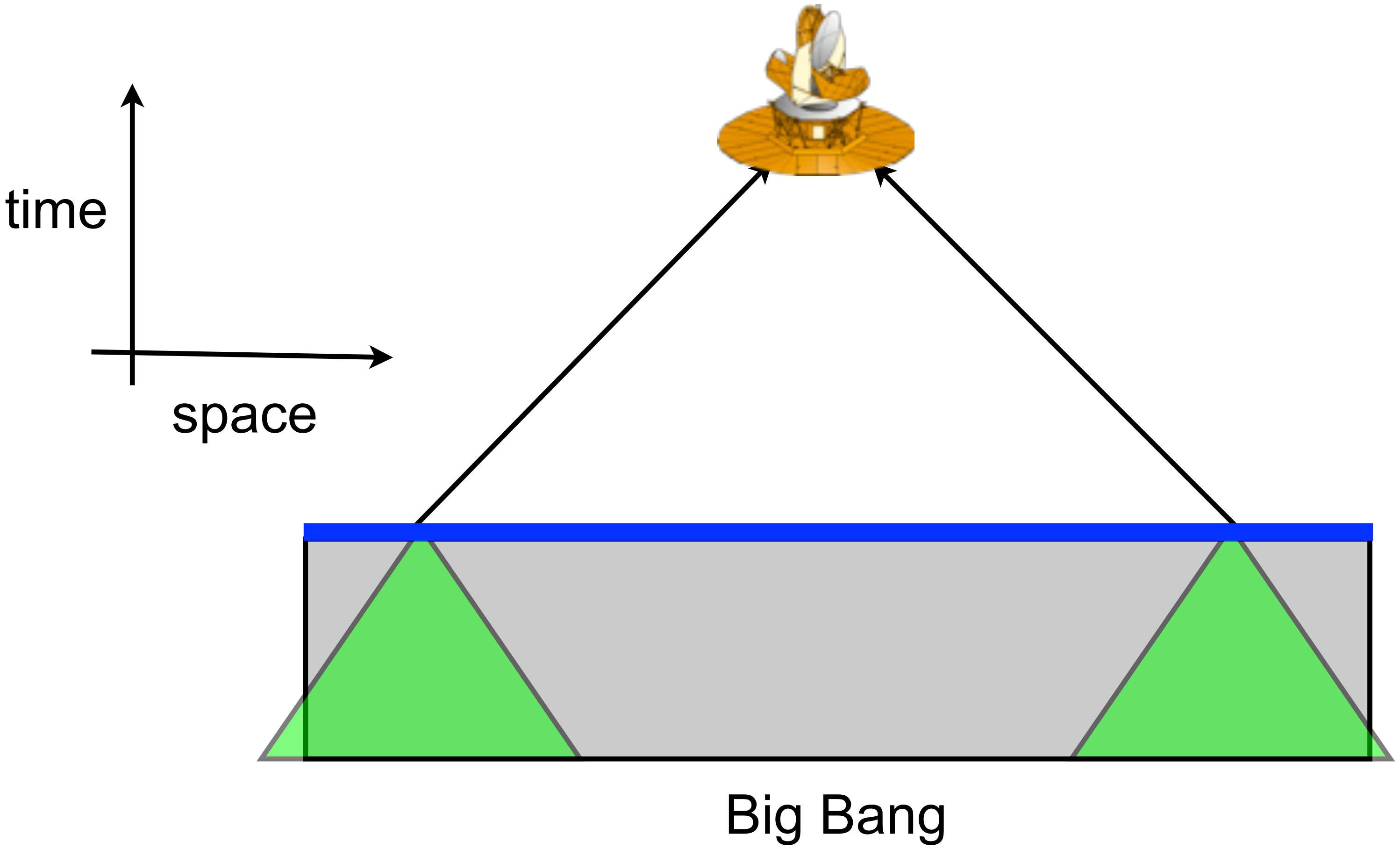
Why?

- Horizon problem: Seemingly acausal correlations.
- Flatness problem: Incredibly finely tuned initial conditions.
- Source of density fluctuations.

Horizon problem



Horizon problem



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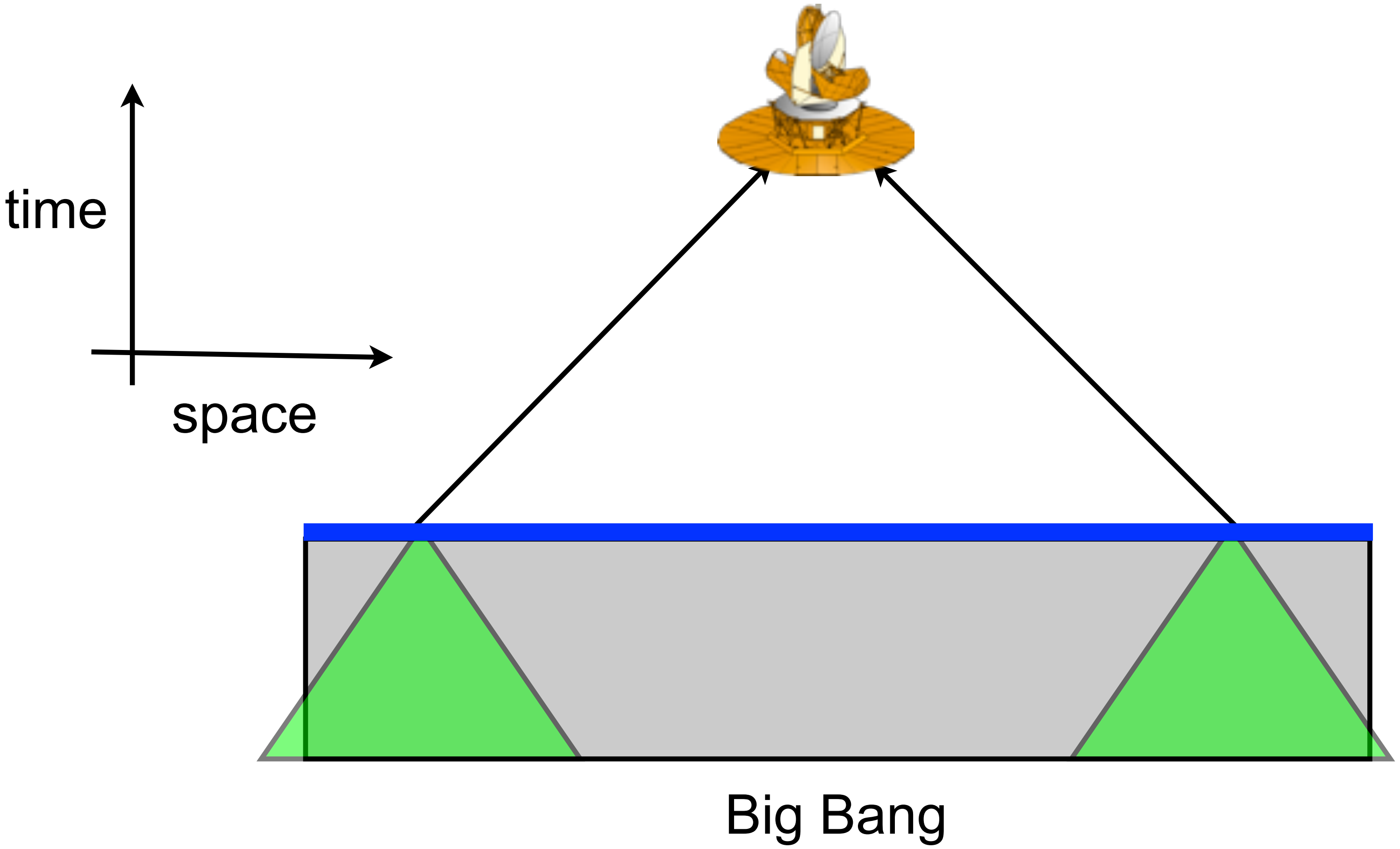


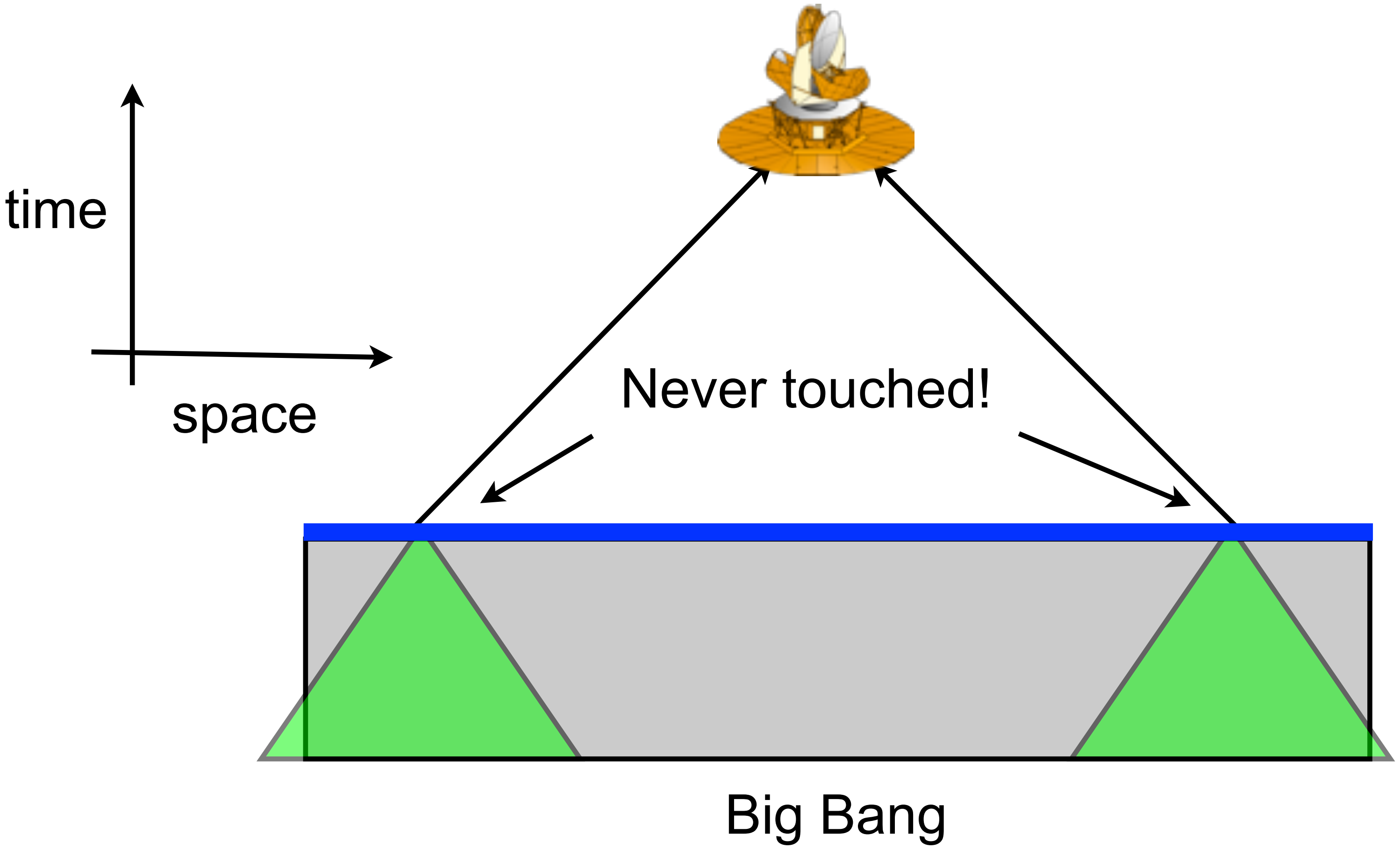
particle horizon

$$\Delta x = \int \frac{dt}{a(t)} = \int_{a=0}^{a_0=1} \frac{d\ln(a)}{aH}$$

comoving horizon

$$\frac{1}{aH} = \eta \qquad \text{(radiation)}$$

Horizon problem



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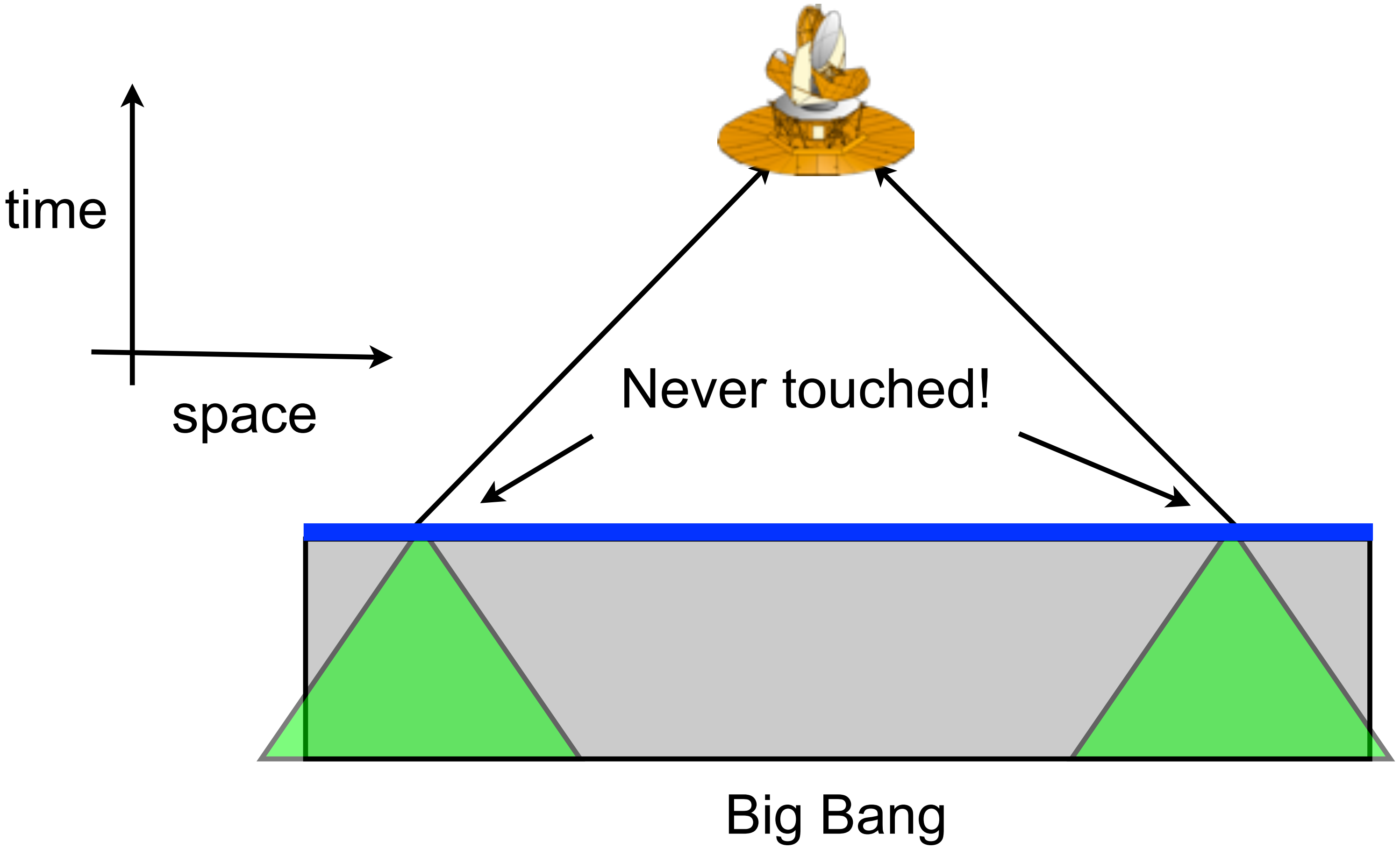

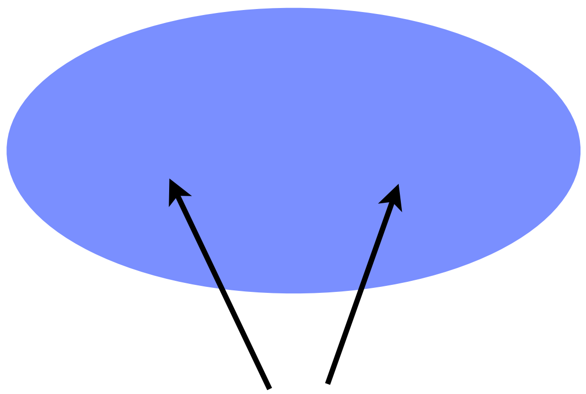

How are these regions at nearly the same temperature?

particle horizon

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Flatness Problem

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{\Omega_k}{a^2}$$

curvature

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- Curvature redshifts slower than matter or radiation.
- The Universe is very nearly flat today, so it must have been extremely flat in the past!

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If the radiation dominated phase
began at the GUT scale:

$$\frac{\Omega_r}{a_I^4} / \frac{\Omega_k}{a_I^2} = \frac{1}{a_I^2}\frac{\Omega_r}{\Omega_k} \sim 10^{60}\frac{\Omega_r}{\Omega_k}$$

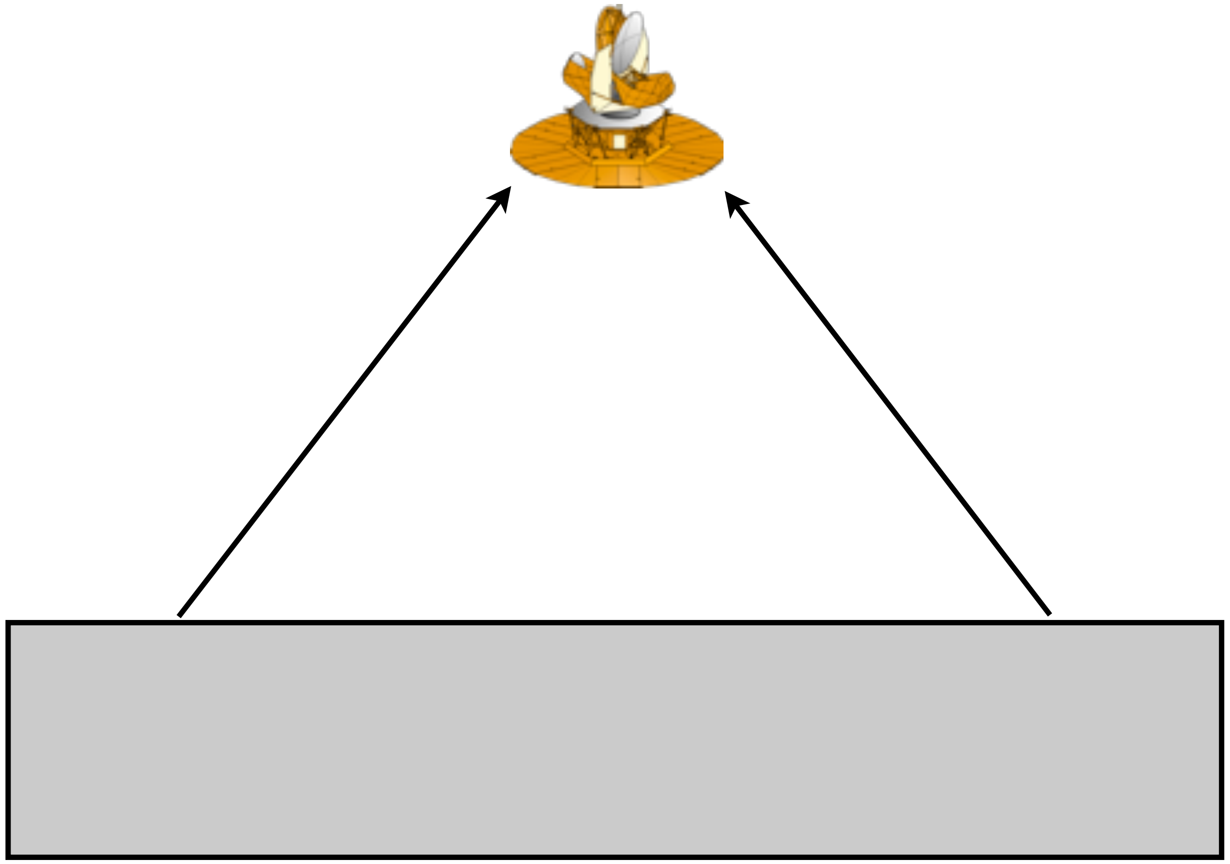
Source of Density Fluctuations

- The primordial spectrum of density fluctuations necessary to explain observations is:
    - Gaussian
    - Nearly scale invariant $n_s \simeq 1$
    - Small amplitude $A \ll 1$
    - Superhorizon
    - Adiabatic

Why this particular set of properties?

$$P(k) = Ak^{n_s - 1}$$

A Solution: Inflation



shrinking
comoving horizon

$$\frac{1}{aH} \sim a^{(1+3w)/2}$$

$$w < -1/3$$

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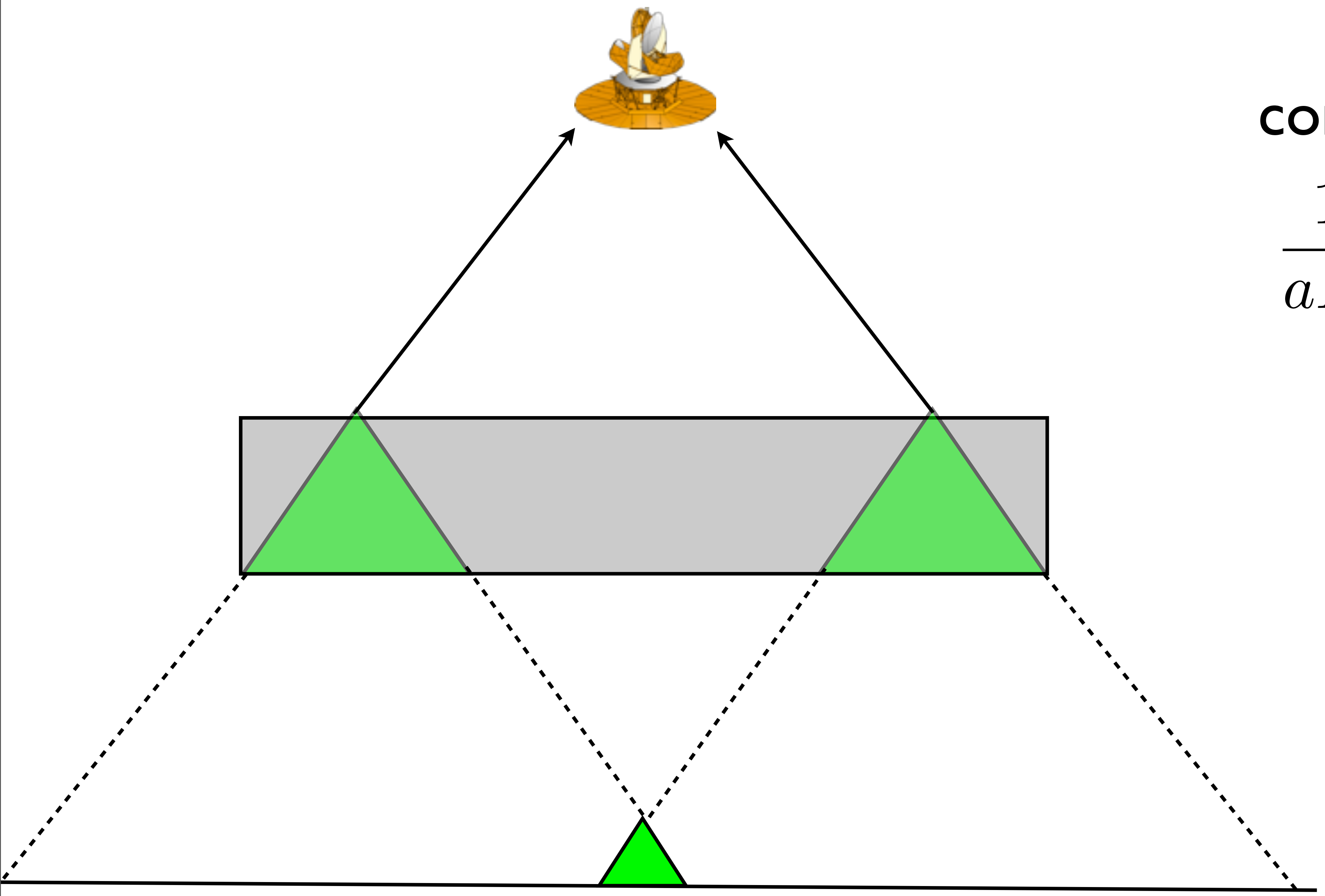

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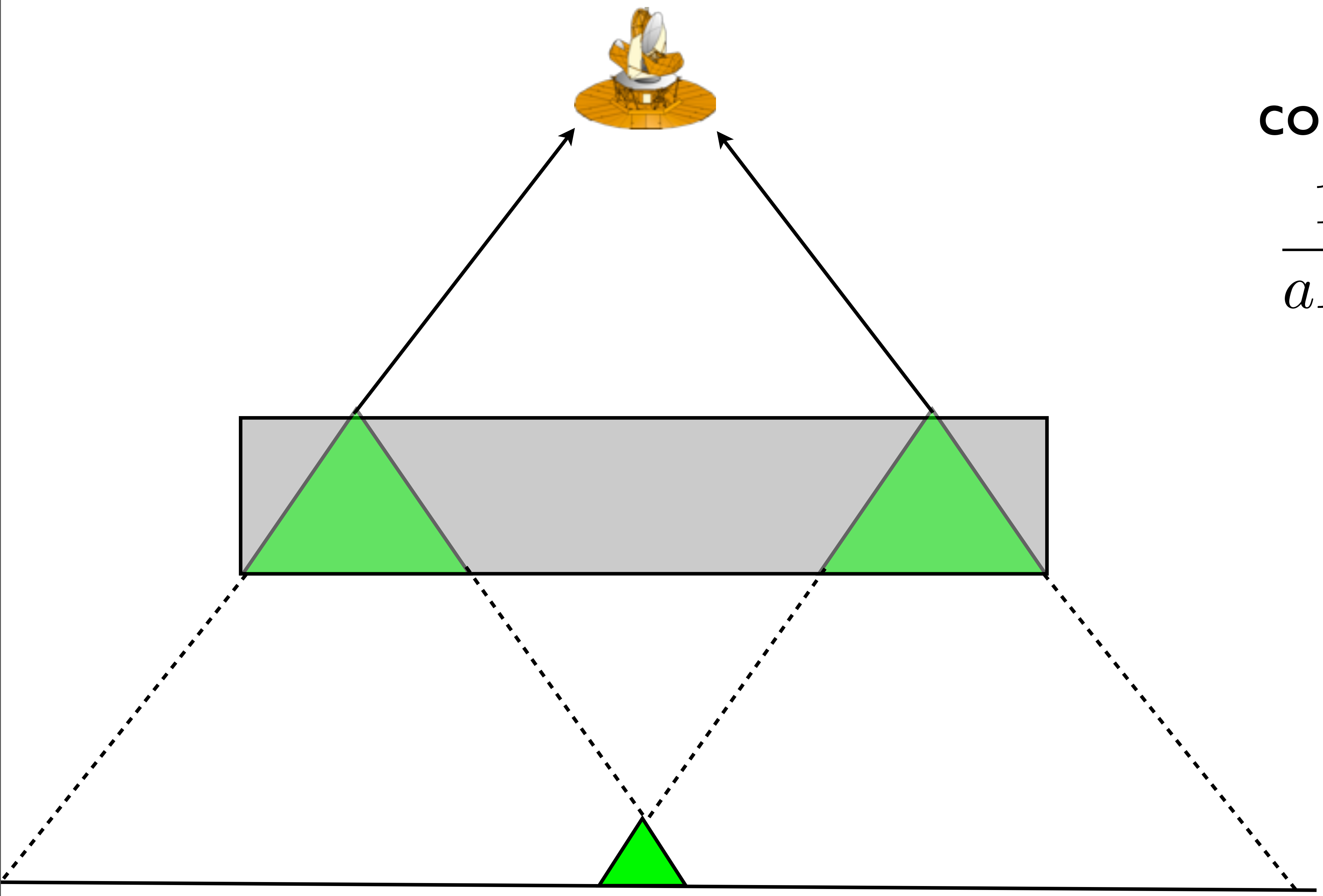

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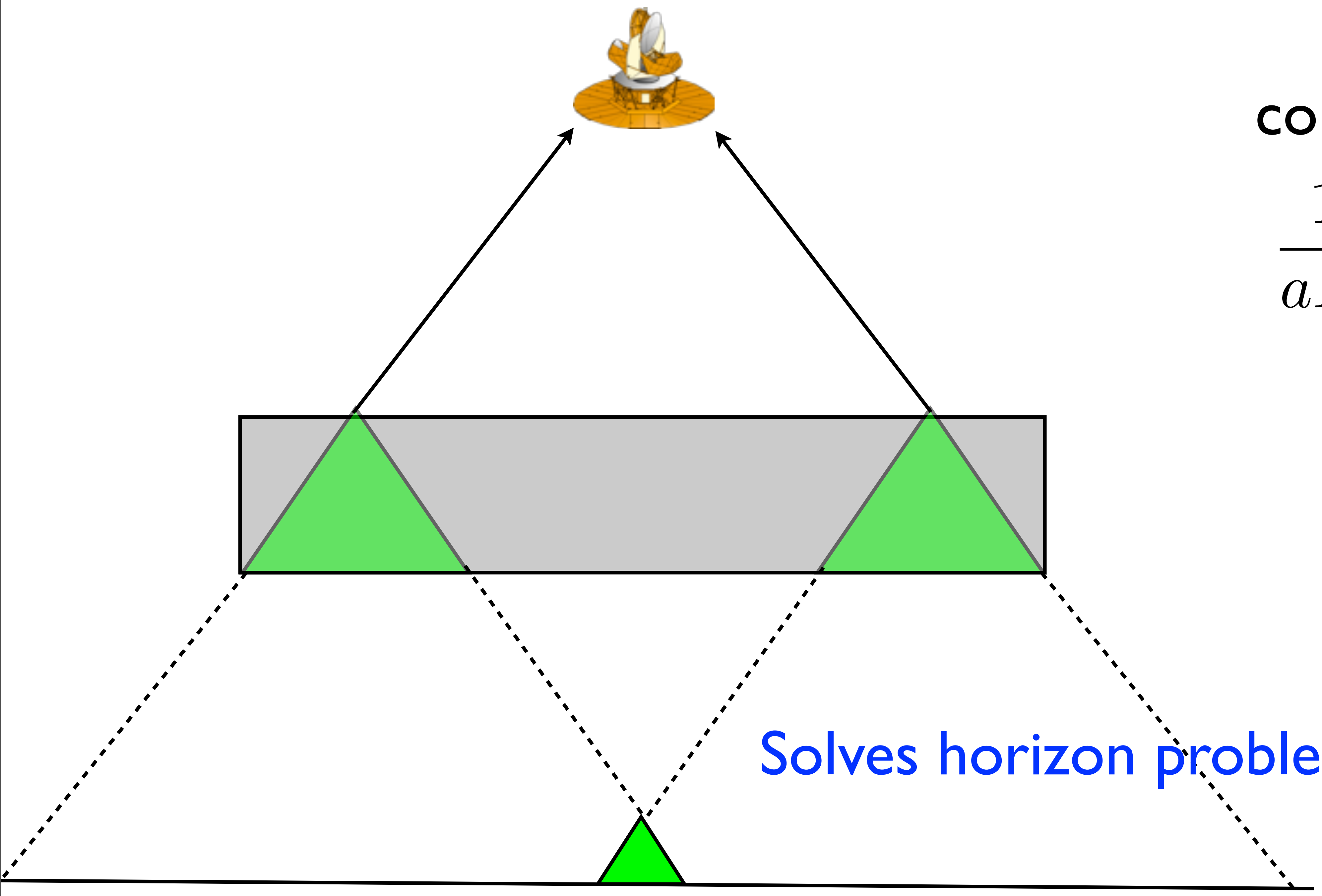

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- $w < -1/3$ also implies:
  - Negative pressure $p_i = w_i \rho_i$
  - Accelerated expansion: $\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(1+3w)a\rho$

A Solution: Inflation

- A de Sitter universe with w=-1 has thermodynamic properties:

$$T = \frac{H}{2\pi} \qquad S = \frac{Area}{4G_N} = \frac{\pi}{H^2 G_N}$$

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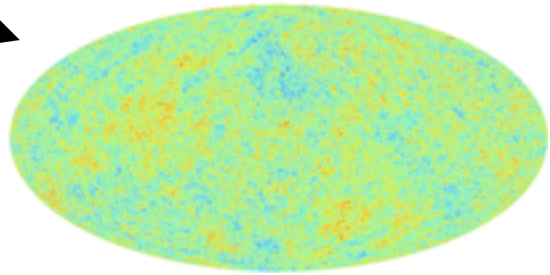

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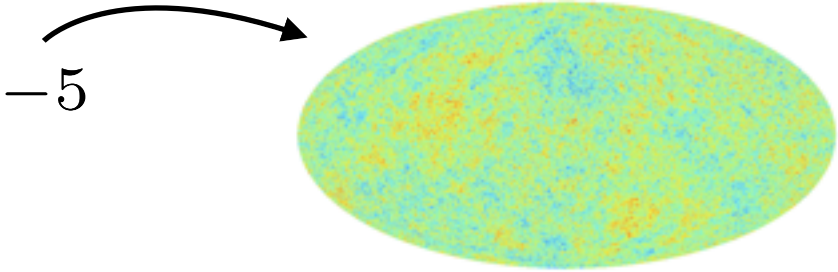

Simplistic picture, but gives flavour of solution.

The Inflaton

- Scalar fields:

$$T^{\mu}{}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi - g^{\mu}{}_{\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi + V(\phi)\right]$$

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kinetic potential

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kinetic potential

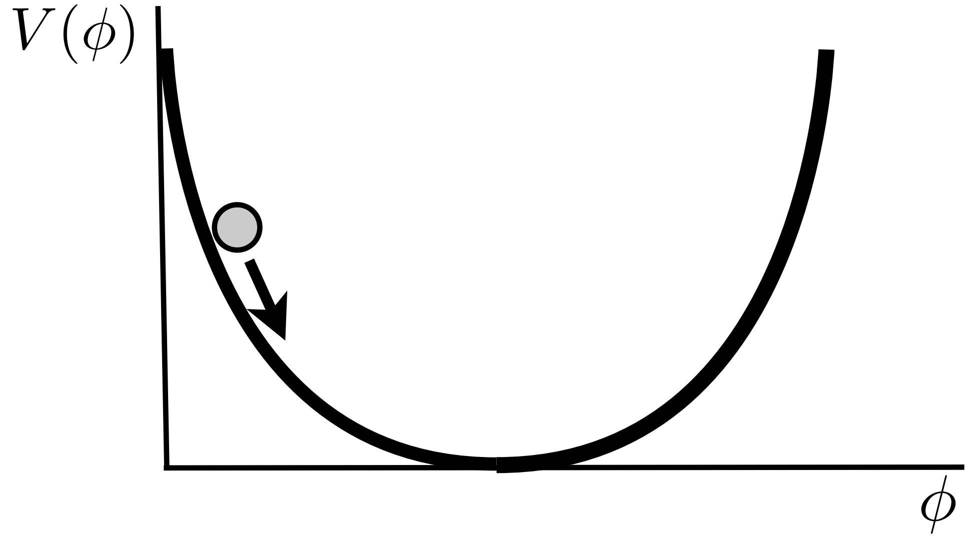
Accelerated expansion results when:

$$V \gg \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 \qquad a \simeq e^{Ht}$$

The Inflaton

- Slow-roll inflation:

$$\frac{d^2\phi}{dt^2} + \underset{\text{friction}}{3H\frac{d\phi}{dt}} = \underset{\text{gradient}}{-\frac{dV}{d\phi}}$$



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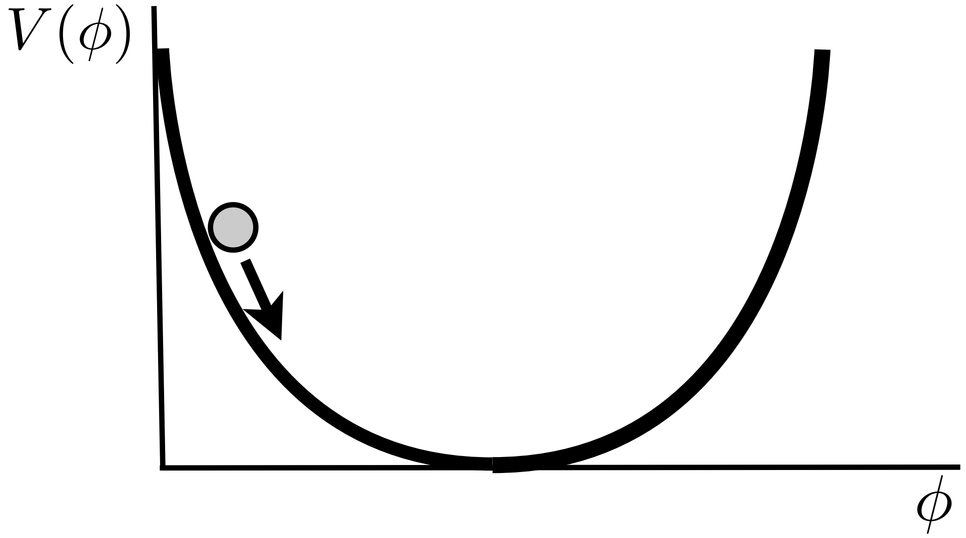

- Friction dominated motion: slow-roll parameters small

$$\epsilon = \frac{M_p^2}{48\pi}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 \qquad \eta = \frac{M_p^2}{24\pi}\frac{1}{V}\left|\frac{d^2V}{d\phi^2}\right|$$

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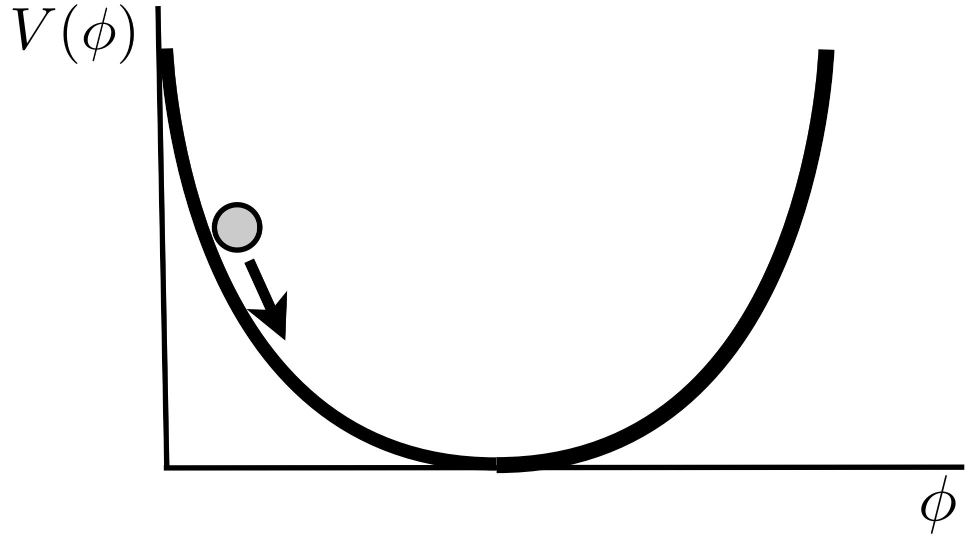

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- Need a sufficiently high and/or flat potential.

Inflaton is an Attractor

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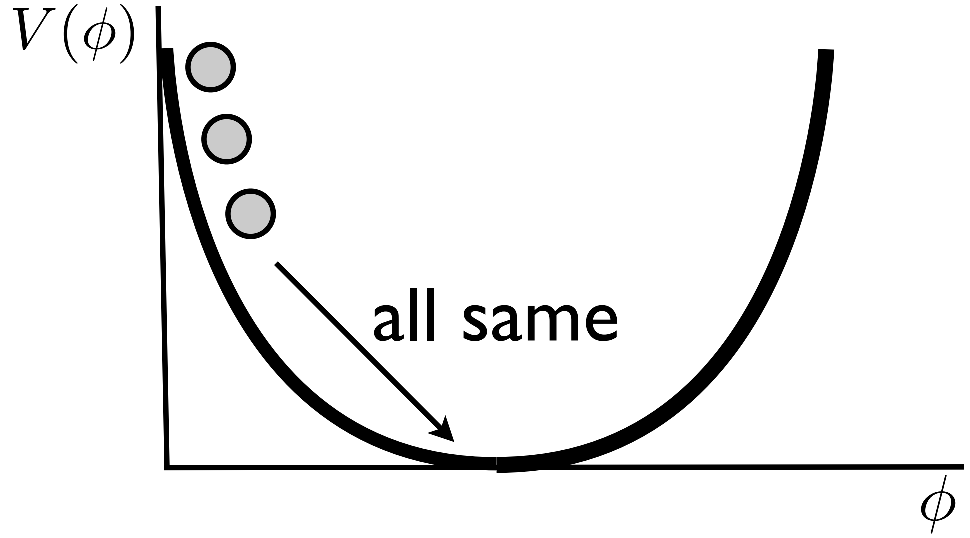

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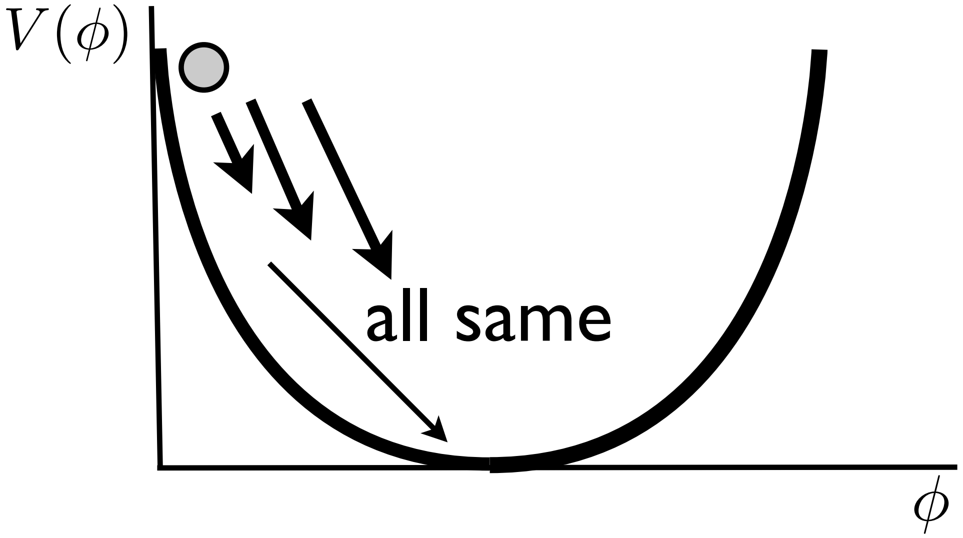

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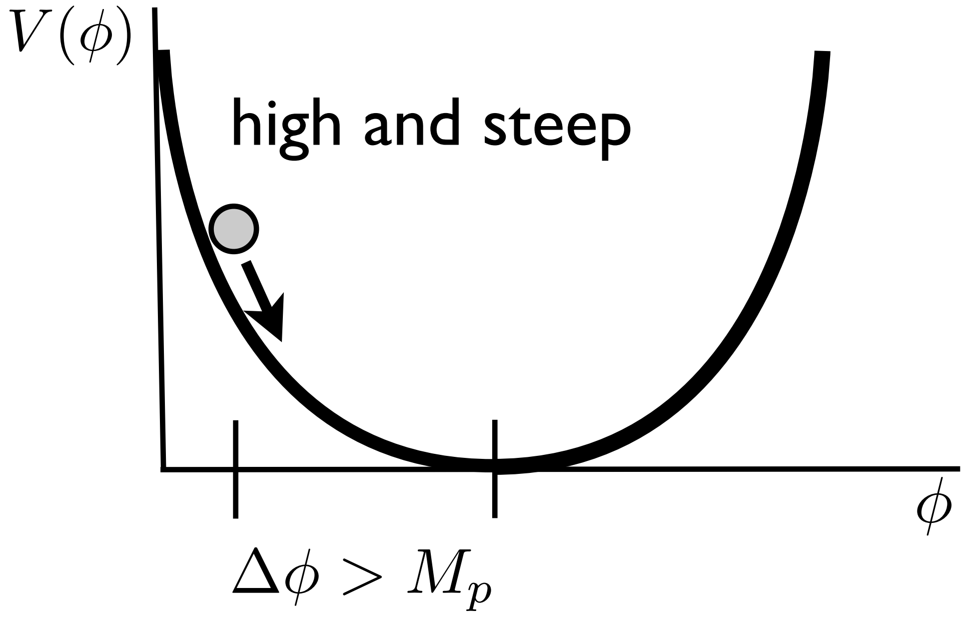
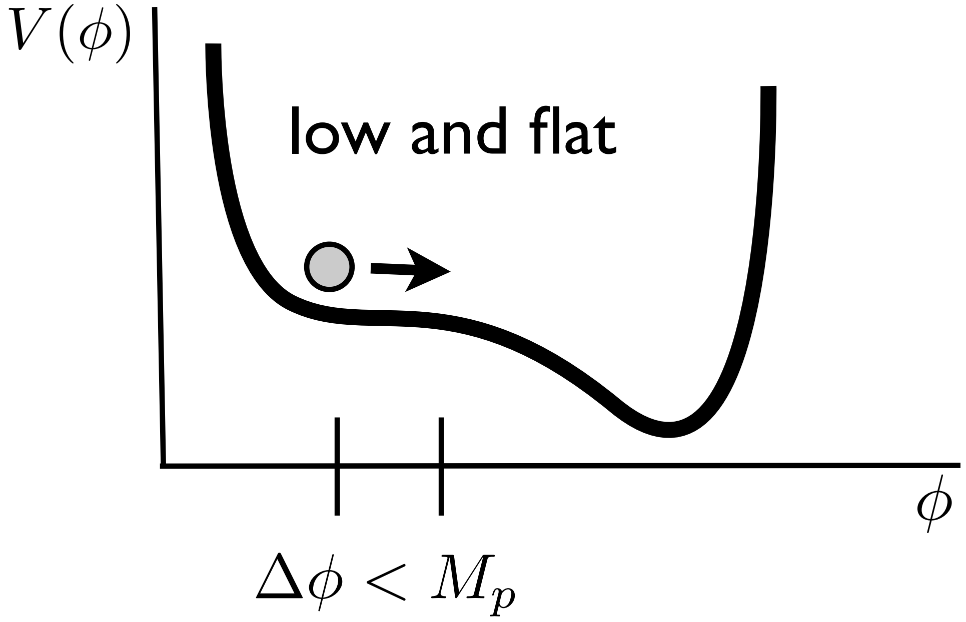
The Inflaton

- Many potentials can drive inflation.



large field



small field

- Differ in the energy scale at which inflation occurs.

The Inflaton

- Where do these potentials come from?
- Inflation is an effective theory: valid below some energy scale.

e.g. Newtonian gravity and GR,
Maxwell and electroweak, etc.

$$V = V_0 \left(1 + c_2 \frac{\phi^2}{M_p^2} + \ldots \right)$$

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- Inflation is an effective theory sensitive to the physics of quantum gravity:

$$\eta = \frac{M_p^2}{24\pi} \frac{1}{V} \left| \frac{d^2 V}{d\phi^2} \right|$$

$$\Delta\eta \sim c_2$$

Why is this correction small?

The Inflaton

- Connection between quantum gravity and inflation is a blessing and a curse.

Blessing:	Curse:
Observational tests of quantum gravity	We don’t have a complete theory of quantum gravity

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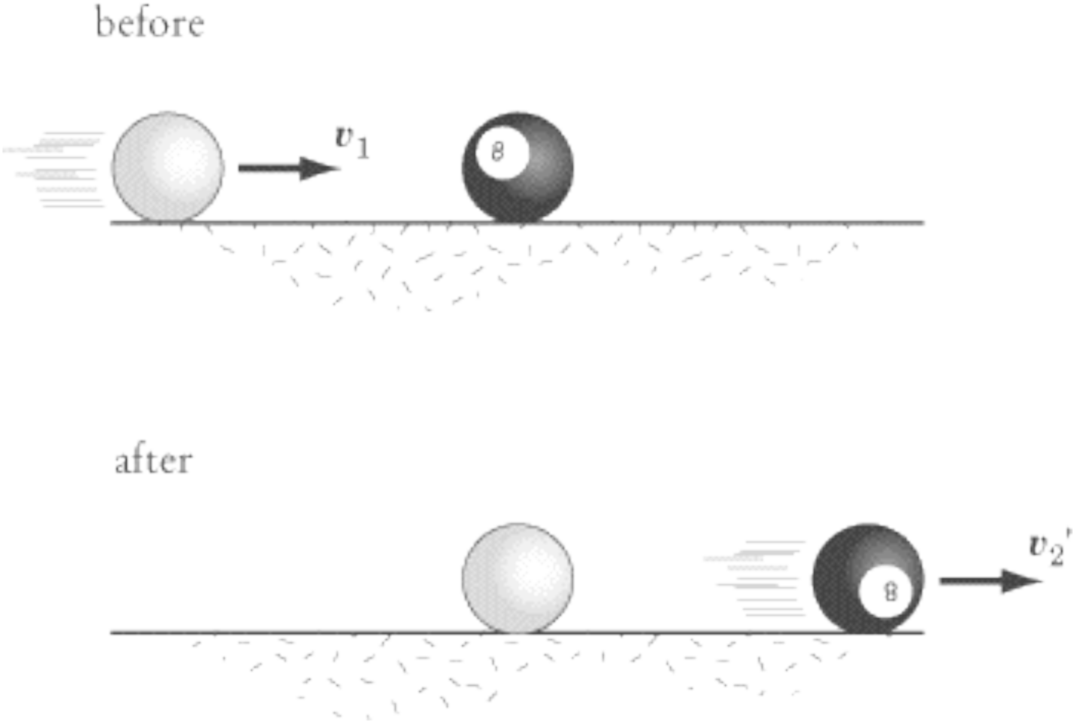
- String theory is the current best candidate theory of quantum gravity $\longrightarrow$ string inflation!

String Theory



Nearly all of modern physics: point particles.

This worked until........

String Theory

(Graviton: supervillain from Marvel Comics)

String Theory

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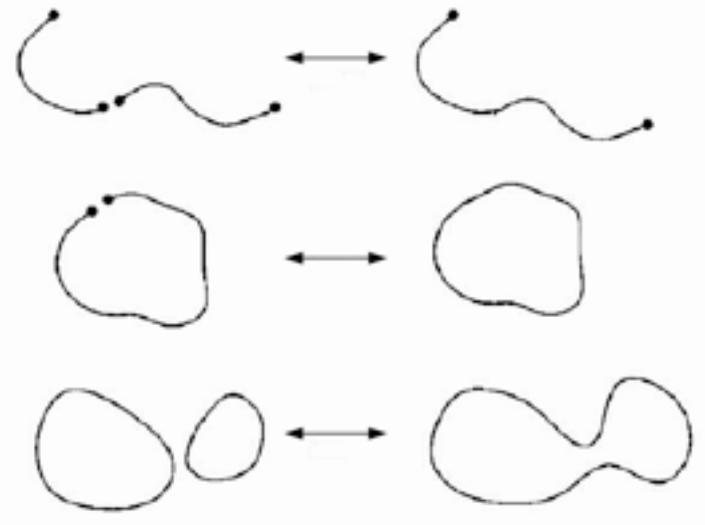
∞ The theory of gravitons does not work! ∞

(not a good quantum theory of gravity)

String Theory



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- Unifies all forces and fundamental particles!

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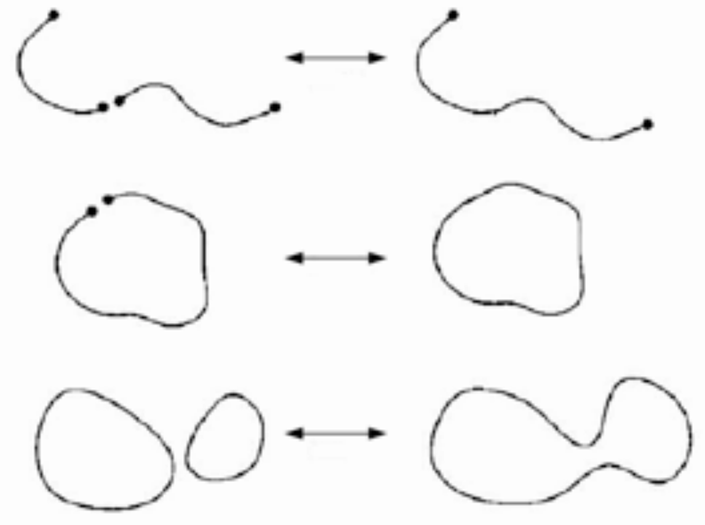

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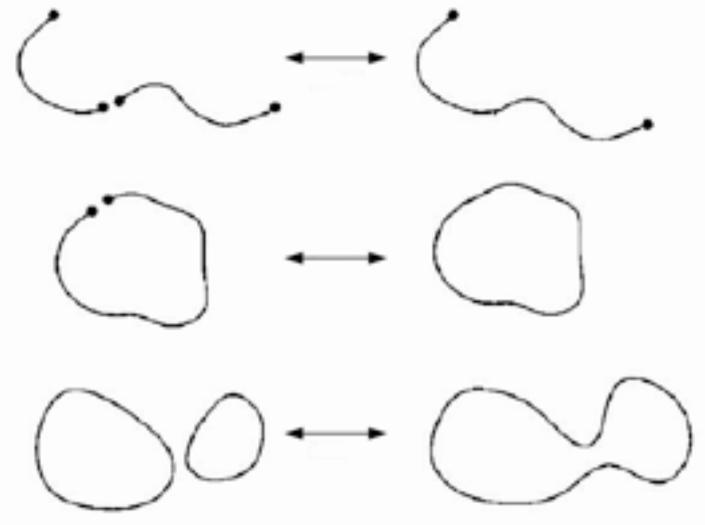

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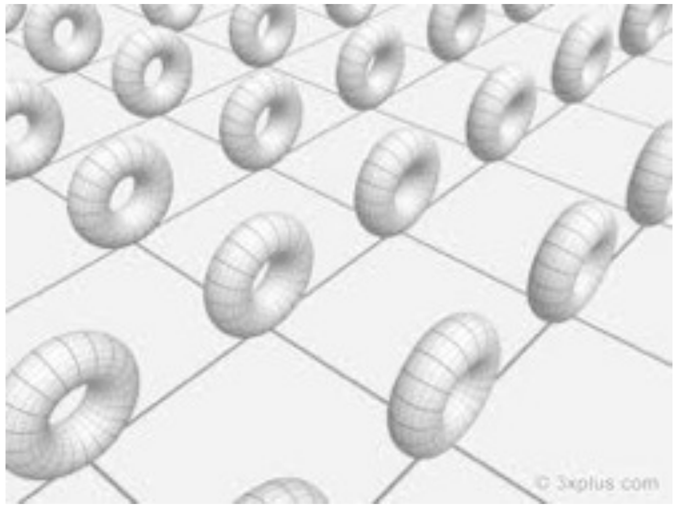

The solution: make the extra dimensions small!

String Theory

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- Changing size changes potential energy: inflation can be driven by the energy stored in the extra dimensions.

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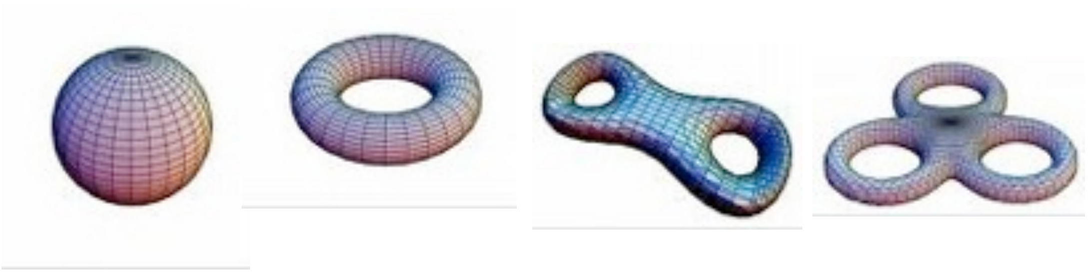

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(Many possible values of the Cosmological Constant)

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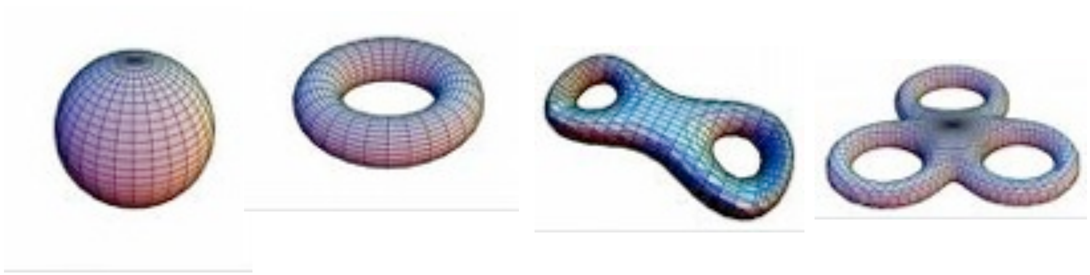

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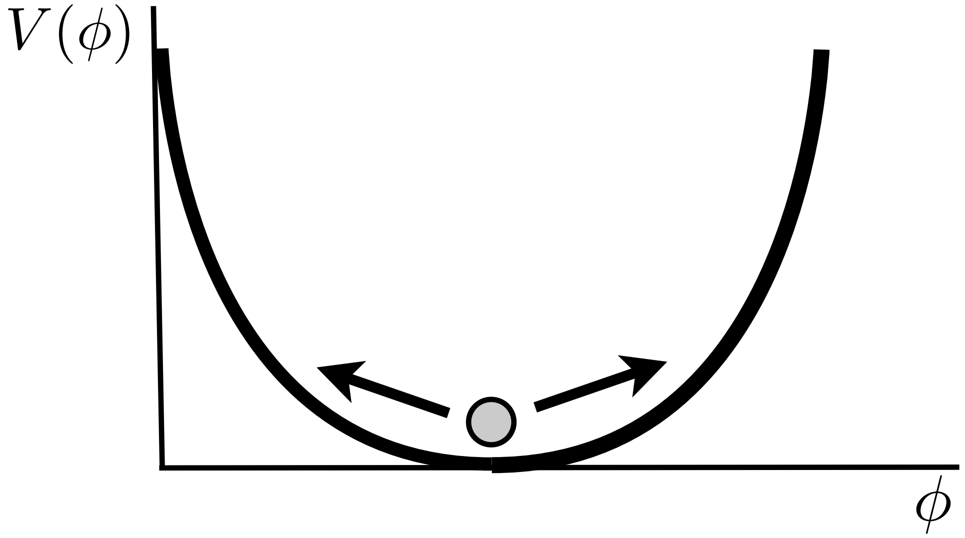
- To do list: how do we make predictions then?

Reheating

- Inflation has to come to an end.



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friction gradient

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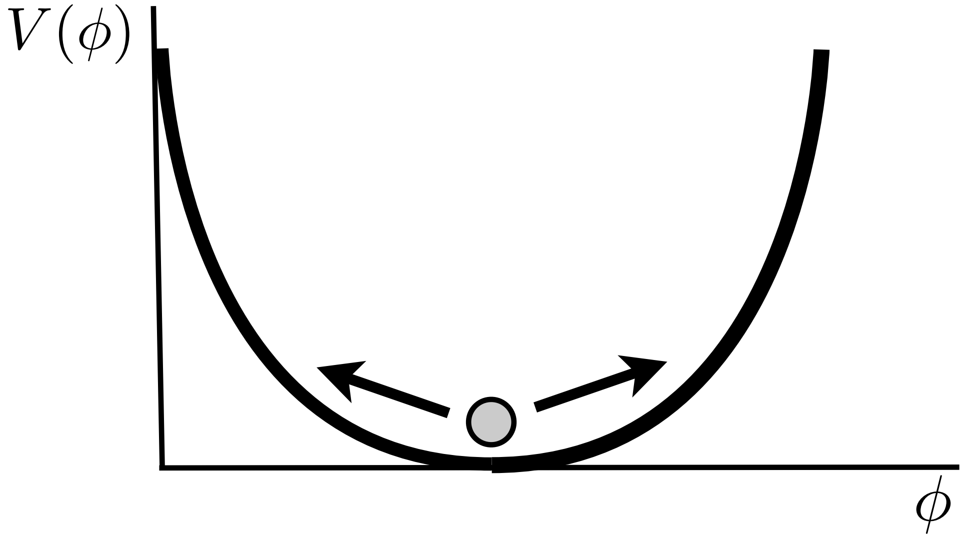

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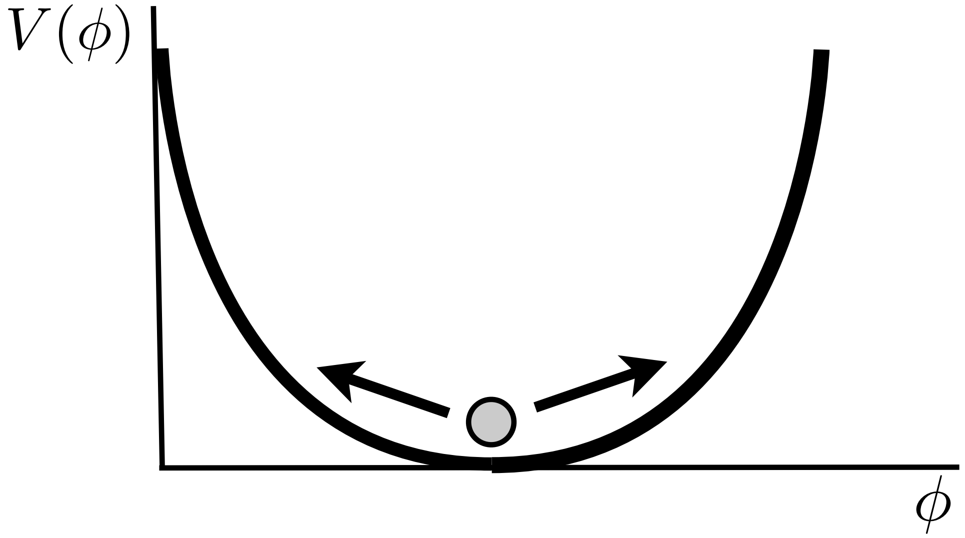

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- The standard story of the hot big-bang follows.

Number of e-folds

- Total expansion of the Universe during inflation:

$$N_e = \log \frac{a_{\rm end}}{a_{\rm begin}} \simeq \frac{M_p}{4\sqrt{3\pi}} \int_{\phi_{\rm end}}^{\phi_{\rm begin}} \frac{d\phi}{\sqrt{\epsilon}}$$

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- To solve the horizon problem, need our observable universe to come from single primordial Hubble patch:

$$\frac{a_0}{a_{\rm begin}} = \frac{a_0}{a_{\rm eq}} \frac{a_{\rm eq}}{a_{\rm reh}} \frac{a_{\rm reh}}{a_{\rm begin}} = \frac{H_{\rm begin}}{H_0} \simeq 10^{55}$$

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$$\uparrow \qquad\qquad\qquad \uparrow$$

$$3000 \, \frac{T_{\rm reh}}{T_{\rm eq}} \qquad e^{N_e}, \quad N_e \sim 60$$

Classical Fields

- Scalar field in Minkowski space:

$$S = \int d^3x dt \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_i \phi)^2 - V(\phi) \right]$$

$$\frac{d^2\phi}{dt^2} - \nabla^2\phi + m^2\phi = 0$$

Classical Fields

- Scalar field in Minkowski space:

$$S = \int d^3x dt \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_i \phi)^2 - V(\phi) \right]$$

$$\frac{d^2\phi}{dt^2} - \nabla^2 \phi + m^2 \phi = 0$$

- Go to fourier space: free field theory is an infinite number of independent oscillators.

$$\frac{d^2\phi_{\vec{k}}}{dt^2} + \omega_{\vec{k}}^2 \phi_{\vec{k}} = 0$$

$$\phi(t,x) = \int \frac{d^3k}{(2\pi)^3}\ \phi_{\vec{k}}(t)\ e^{i\vec{k}\cdot\vec{x}}$$

$$\omega_{\vec{k}}^2 = k^2 + m^2$$

Quantum Fields

- Promote fields to operators:

$$\phi \to \hat{\phi} \quad \pi \to \hat{\pi}$$

$$[\hat{\phi}(t,\vec{x}),\hat{\pi}(t,\vec{y})] = i\delta(\vec{x}-\vec{y})$$

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- In fourier space: quantize the infinite number of independent oscillators:

$$\hat{\phi} = \frac{1}{\sqrt{2}}\int \frac{d^3k}{(2\pi)^3}\left[a^-_{\vec{k}} v^*_k e^{i\vec{k}\cdot\vec{x}} + a^+_{\vec{k}} v_k e^{-i\vec{k}\cdot\vec{x}}\right]$$

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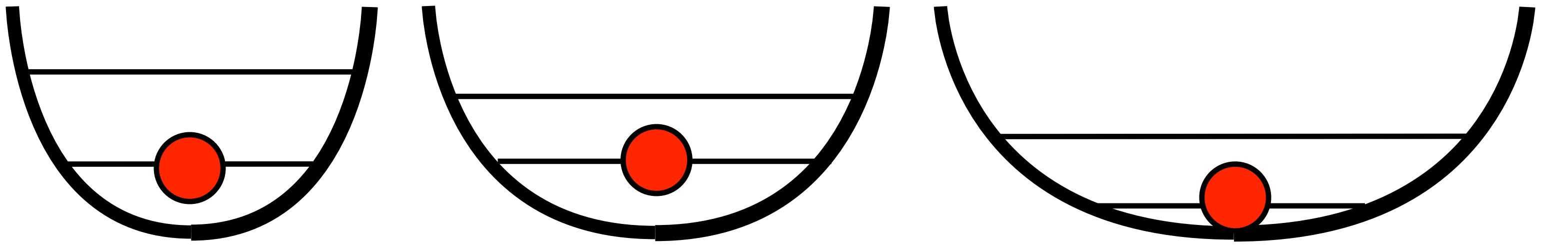
$$v_k = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k t} \qquad [a^-_{\vec{k}}, a^+_{\vec{k}'}] = \delta^3(\vec{k}-\vec{k}')$$

mode function

creation/annihilation operators

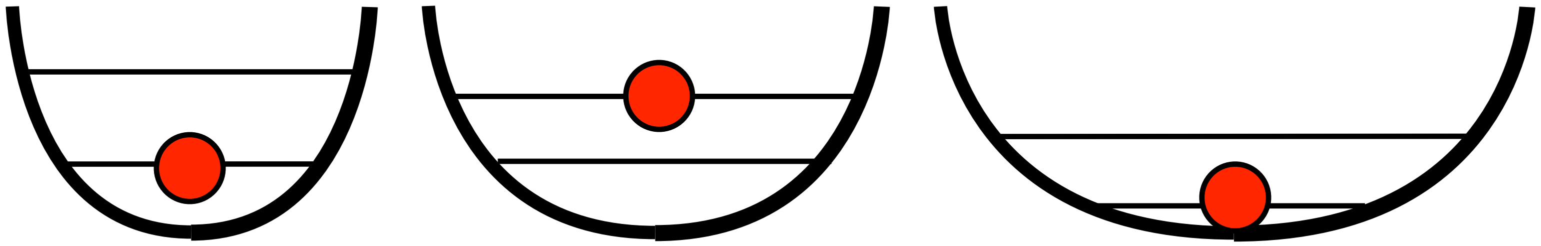
Quantum Fields



$$a^{-}_{\vec{k}}\,|0\rangle = 0$$

vacuum

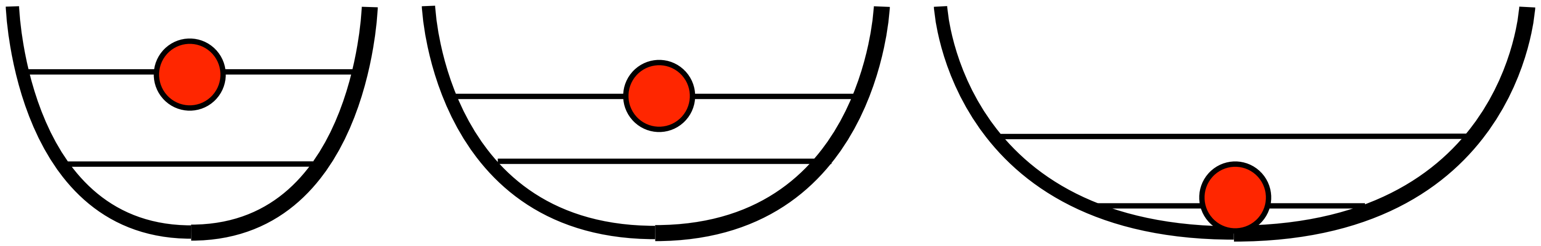
Quantum Fields



$$a^{+}_{\vec{k}}|0\rangle = |1\rangle_{\vec{k}}$$

particle = positive frequency excitation

Quantum Fields



$$a^{+}_{\vec{k}} a^{+}_{\vec{k}'} |0\rangle = |1\rangle_{\vec{k}} |1\rangle_{\vec{k}'}$$

multi-particle states

QFT in Curved Spacetime

- In curved space:

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + m^2\phi\right]$$

QFT in Curved Spacetime

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- Re-cast as canonical free scalar (for FRW):

$$\chi = a\phi \qquad \frac{d^2\chi_{\vec{k}}}{d\eta^2} + \omega^2_{\vec{k}}(\eta)\chi_{\vec{k}} = 0$$

comoving k!

QFT in Curved Spacetime

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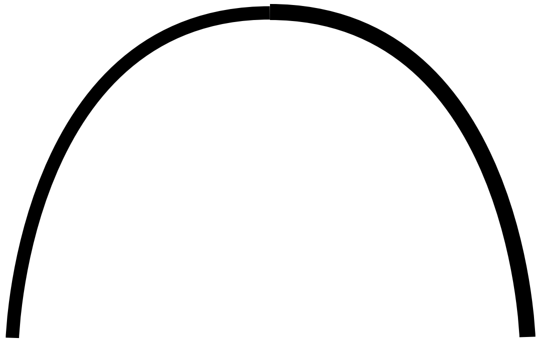
comoving k!

- In de Sitter, the scale factor is: $a = -\frac{1}{H\eta} \qquad -\infty < \eta \leq 0$

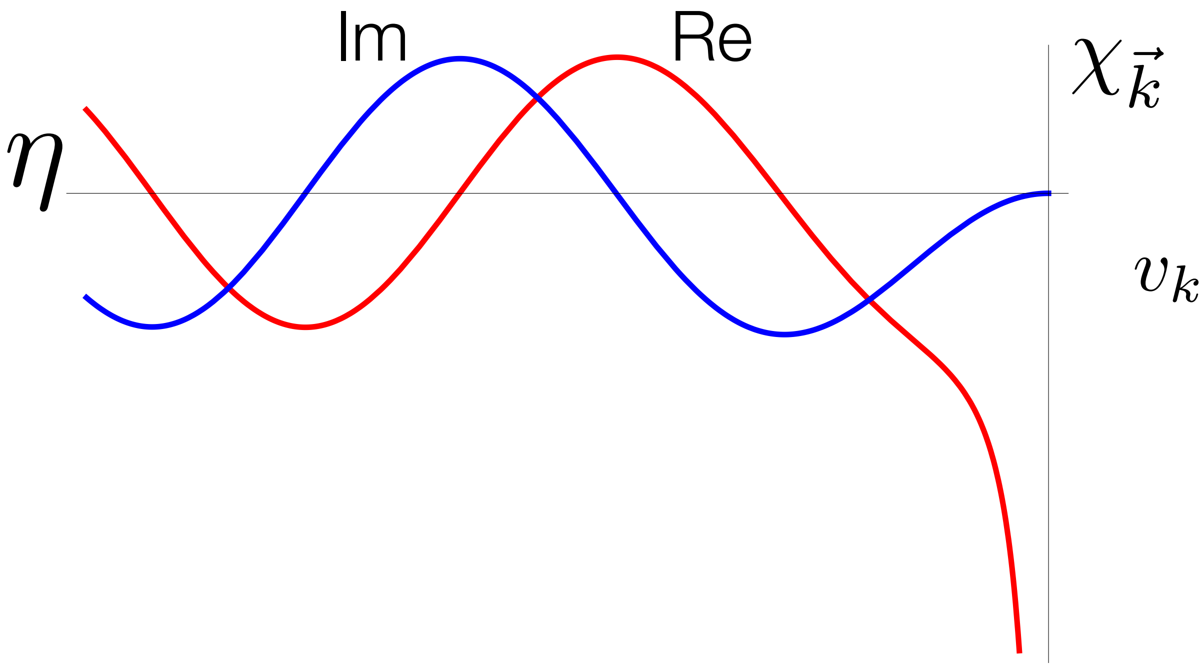
$$\omega^2_{\vec{k}}(\eta) = k^2_{\rm com} + \frac{1}{\eta^2}\left(\frac{m^2}{H^2} - 2\right)$$



$$\omega^2_{\vec{k}}(\eta) < 0, \quad k\eta \ll 1, \quad m \ll H$$

QFT in Curved Spacetime

- Classical solutions to the equation of motion:



$$v_k = \sqrt{\frac{\pi|\eta|}{2}}\left[J_n(k\eta) - iY_n(k\eta)\right]$$

$$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

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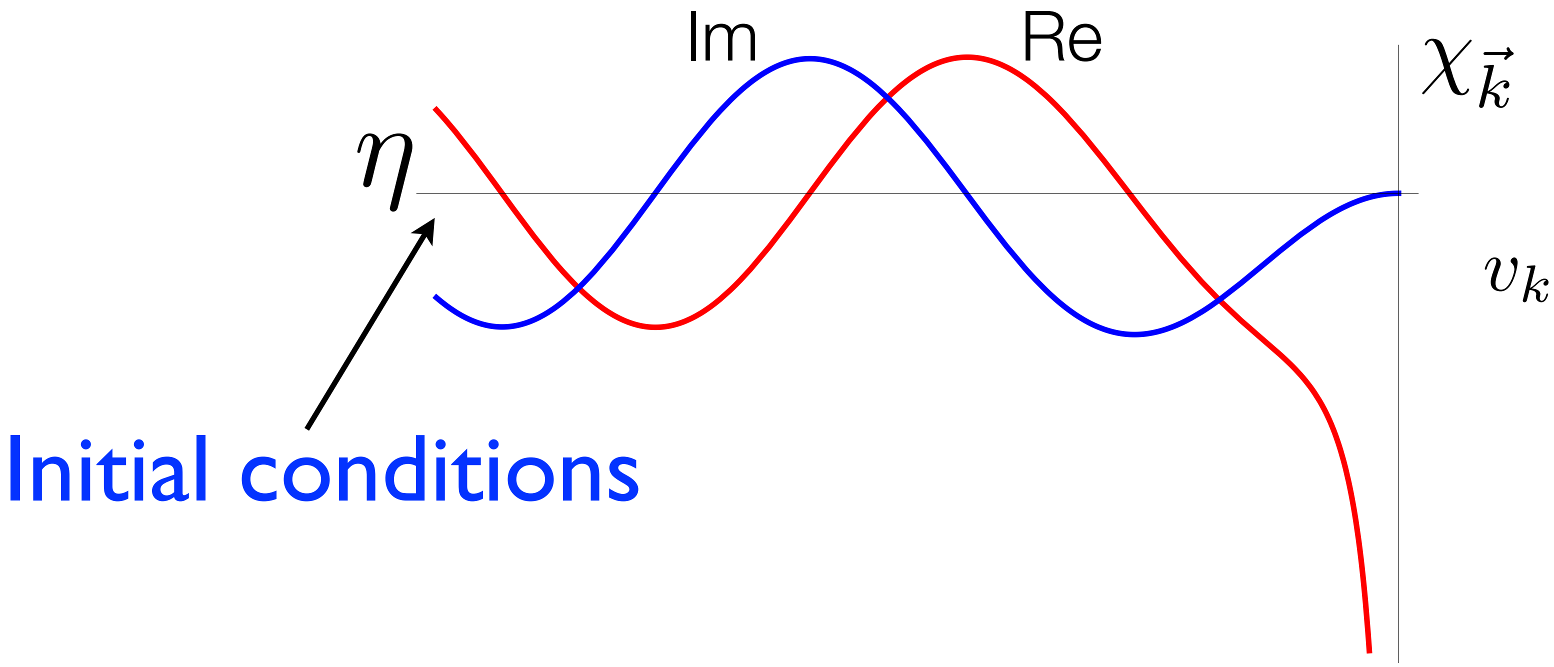

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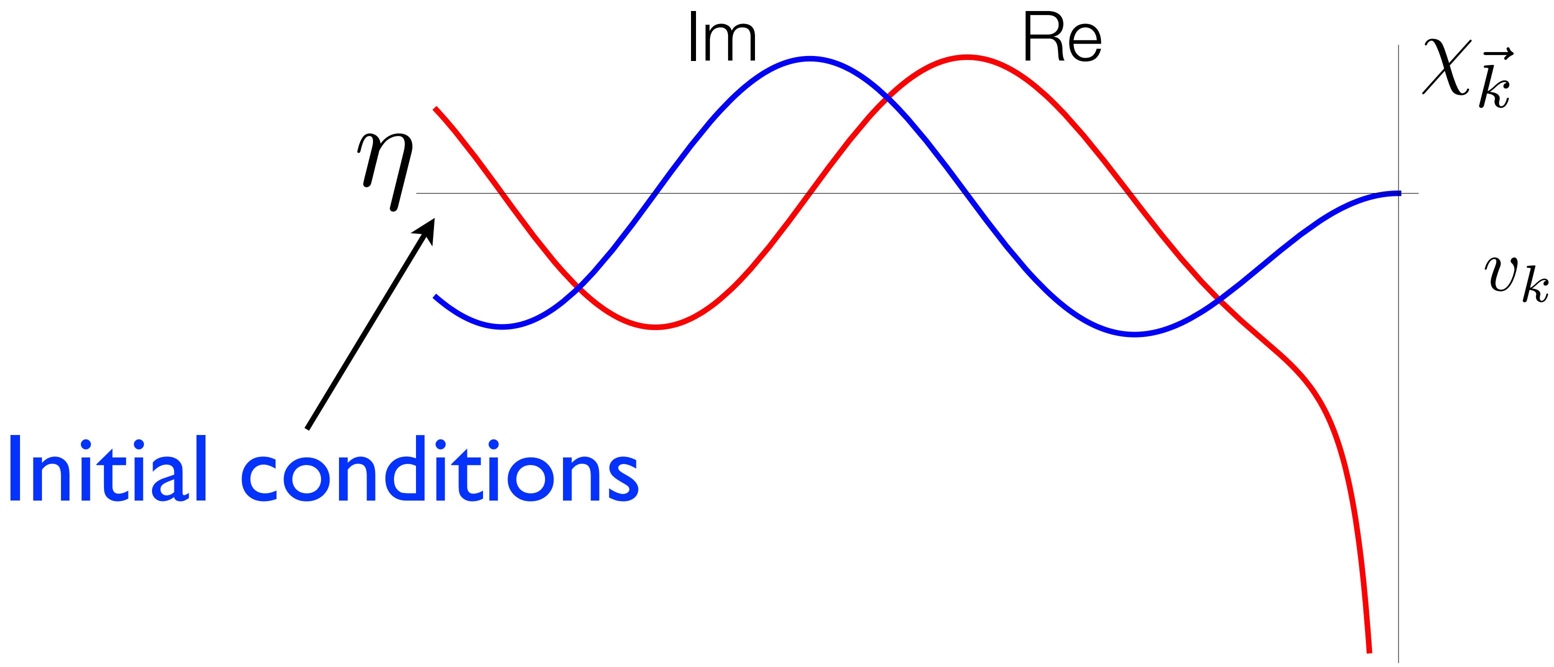

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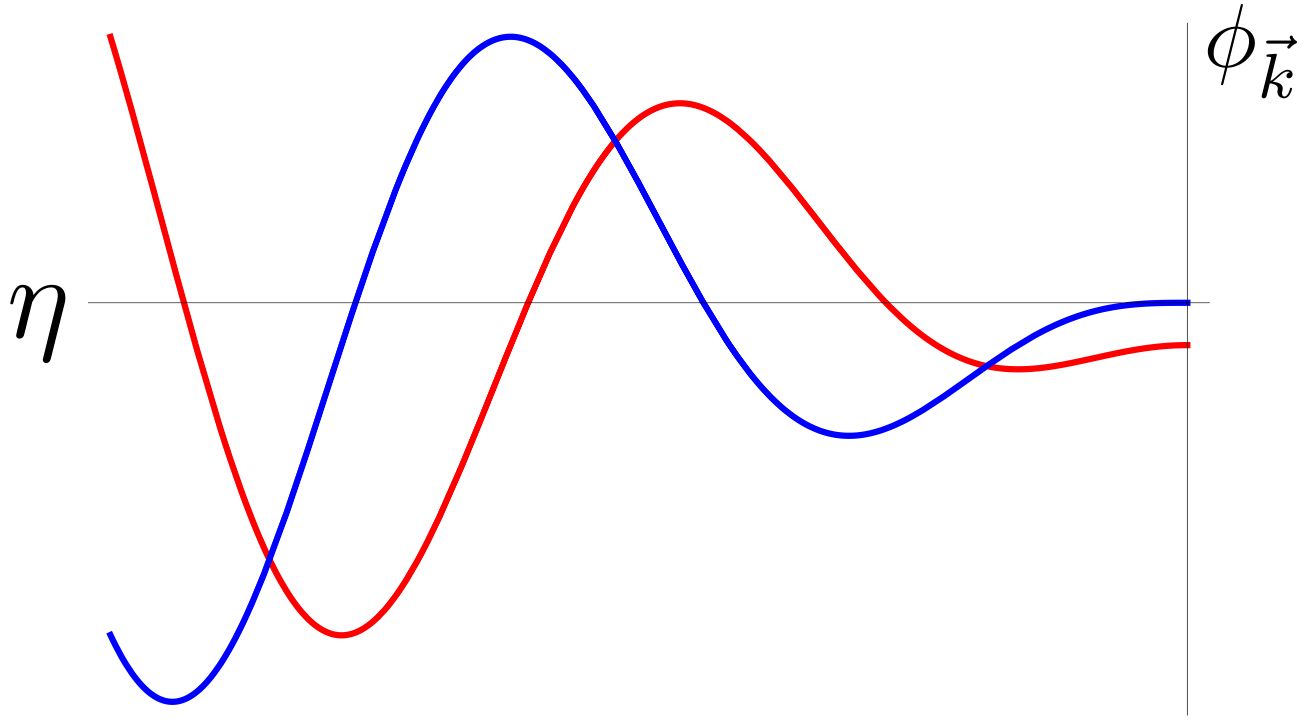

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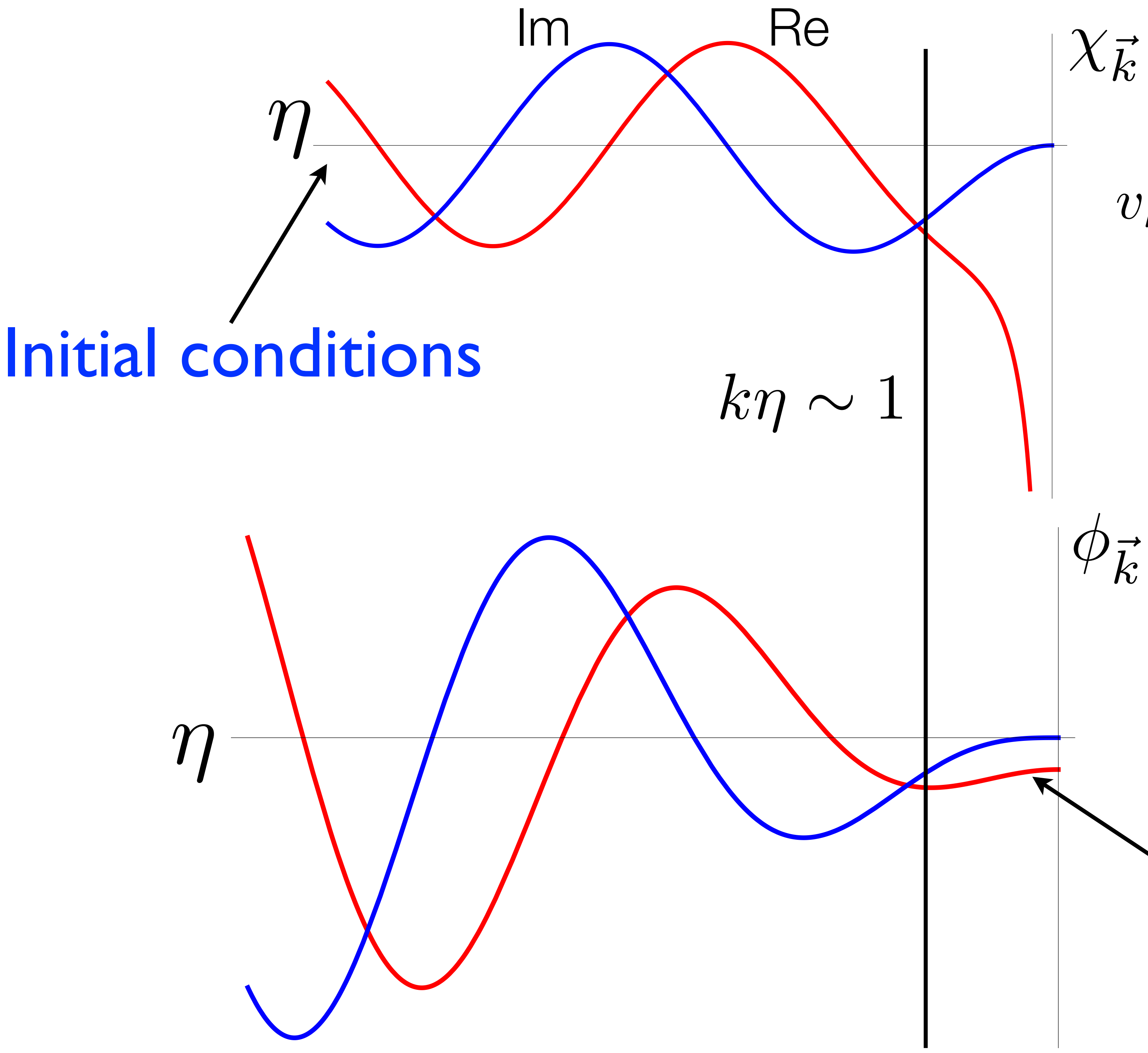

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$$n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

The field ``freezes in''

QFT in Curved Spacetime

- Quantize just as before, but.....

$$\omega_{\vec{k}}^2(\eta) = k_{\rm com}^2 + \frac{1}{\eta^2}\left(\frac{m^2}{H^2} - 2\right)$$

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- A prescription for the vacuum: Minkowski at small scales.

$$v_k = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k \eta}, \quad k\eta \to -\infty$$

Bunch-Davies

QFT in Curved Spacetime

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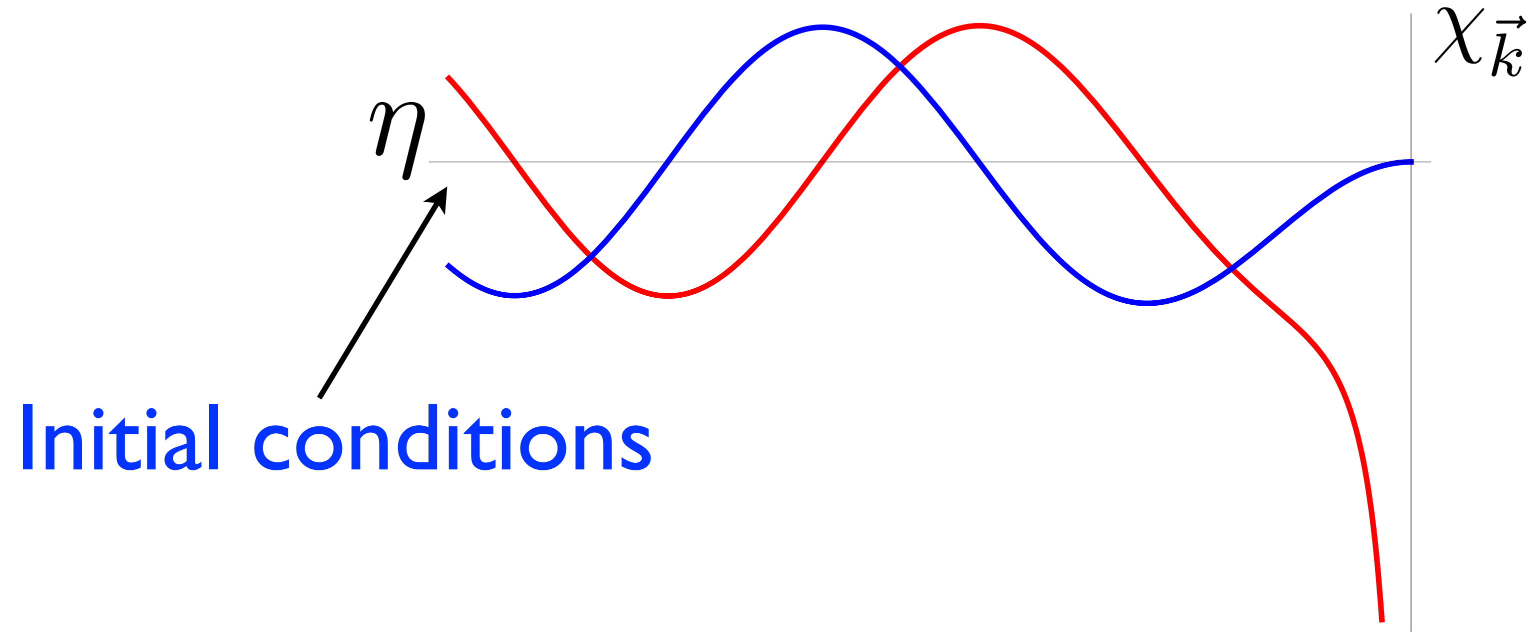

Initial conditions

QFT in Curved Spacetime

$$v_k = \sqrt{\frac{\pi|\eta|}{2}}\,[J_n(k\eta) - iY_n(k\eta)] \qquad |v_k|^2 = \frac{1}{k} + \frac{1}{\eta^2 k^3}$$ (massless)

- Find the correlation functions:

QFT in Curved Spacetime

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Fourier space: $\langle 0|\hat{\chi}_{\vec{k}}\hat{\chi}^*_{\vec{k}'}|0\rangle = \delta(\vec{k} - \vec{k}')\frac{|v_k|^2}{2}$

Real space: $\langle 0|\hat{\chi}(x,t)\hat{\chi}^*(y,t)|0\rangle = \int_0^\infty \frac{dk}{(2\pi)^2}k^2|v_k|^2\frac{\sin(kL)}{kL}$

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Coincident limit:

$$\langle 0|\hat{\chi}(x,t)\hat{\chi}^*(x,t)|0\rangle = \int_0^\infty \frac{dk}{(2\pi)^2}k^2|v_k|^2 \simeq \int_0^{\tilde{k}} \frac{dk}{(2\pi)^2}\frac{1}{\eta^2 k} + \int_{\tilde{k}}^\infty \frac{dk}{(2\pi)^2}k$$

IR UV

QFT in Curved Spacetime

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$$\langle 0|\hat{\phi}(x,\eta)\hat{\phi}^*(x,\eta)|0\rangle = \frac{1}{a^2}\langle 0|\hat{\chi}(x,\eta)\hat{\chi}^*(x,\eta)|0\rangle = \left(\frac{H}{2\pi}\right)^2 \int_0^{\tilde{k}} \frac{dk}{k}$$

QFT in Curved Spacetime

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- Assume inflation has a finite duration, count modes larger than the comoving horizon, go back to proper time:

$$\langle 0|\hat{\phi}(x,t)\hat{\phi}^*(x,t)|0\rangle = \frac{H^3}{4\pi^2}(t-t_0)$$

- Diverges with increasing time -- pile-up of superhorizon modes. Regulated for non-zero mass.

QFT in Curved Spacetime

Fourier space: $\langle 0|\hat{\chi}_{\vec{k}}\hat{\chi}^*_{\vec{k}'}|0\rangle = \delta(\vec{k}-\vec{k}')\frac{(2\pi)^2}{k^3}P_\chi(k)$

- Transform back to the original field:

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- The power spectrum of a free field in dS is:
    - Nearly scale invariant
    - Gaussian uncoupled harmonic oscillators!
    - Small amplitude (compared to...)

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!!Gravitational waves!!

Quantum to Classical

$$\langle 0|\hat{\phi}(x,\eta)\hat{\phi}^*(y,\eta)|0\rangle \rightarrow \langle\hat{\phi}(x,\eta)\hat{\phi}^*(y,\eta)\rangle$$

Quantum expectation value	Ensemble average Spatial average

???

(pure dS might not be best example...)

Inflationary Fluctuations

- The field couples to the metric - can choose a convenient coordinate system (gauge).

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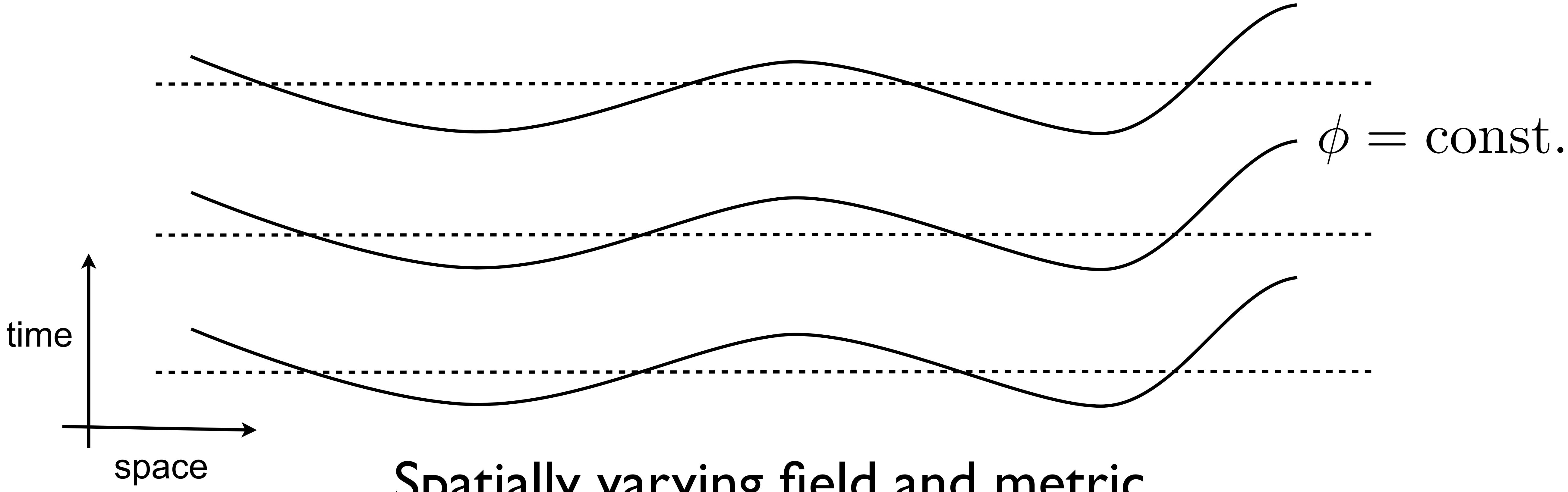

Spatially varying field and metric.

Inflationary Fluctuations

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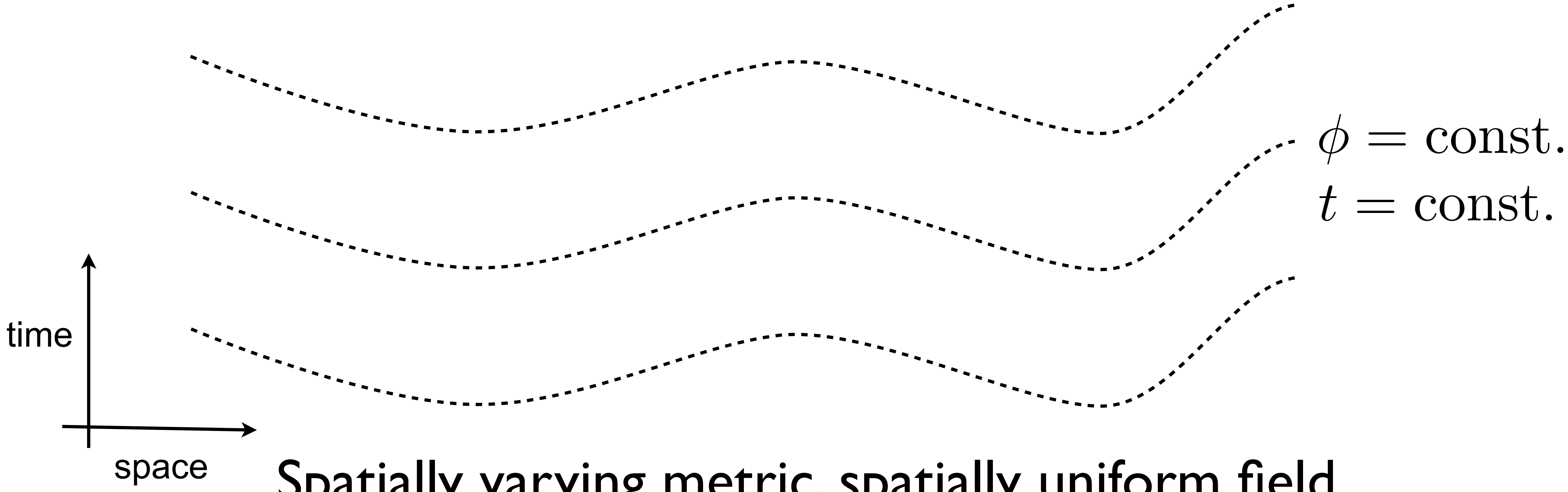

Spatially varying metric, spatially uniform field.

Comoving curvature perturbation: $\mathcal{R}$

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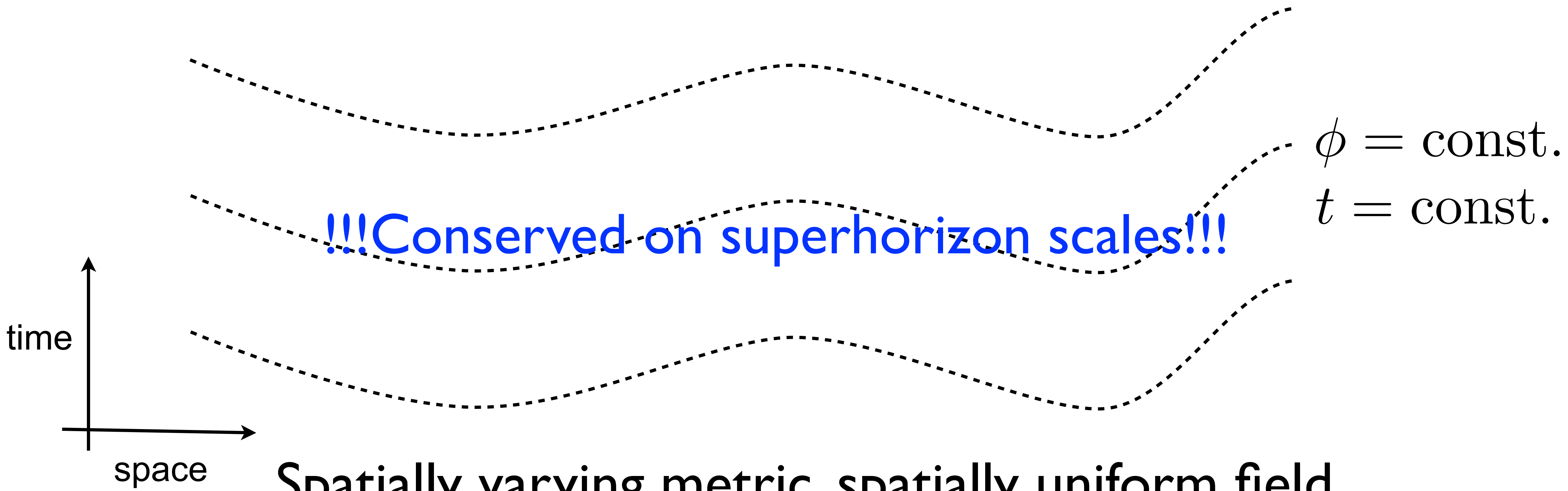


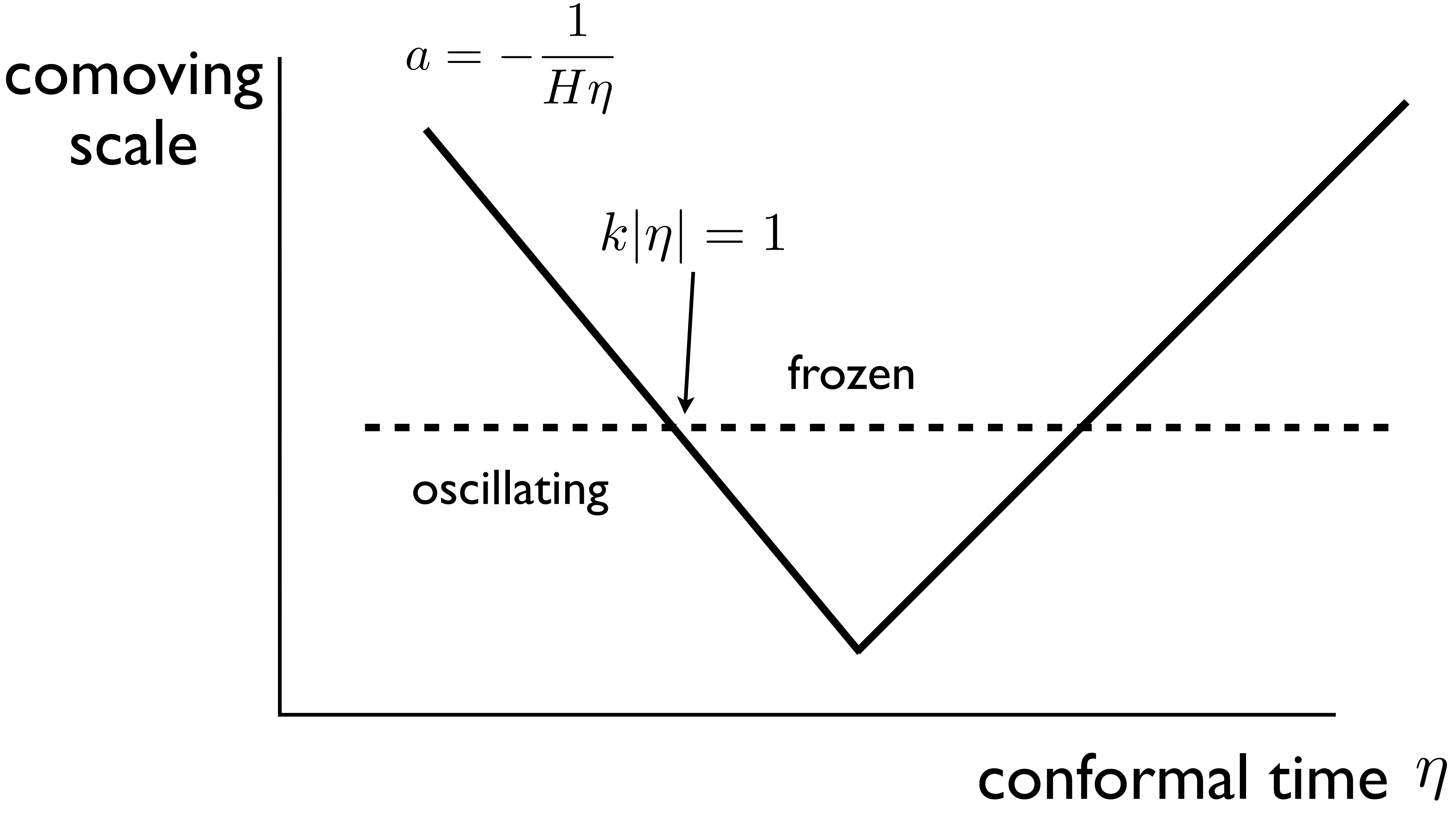
!!!Conserved on superhorizon scales!!!

Spatially varying metric, spatially uniform field.

Comoving curvature perturbation: $\mathcal{R}$

QFT in Curved Spacetime

- An important scale: comoving horizon $\frac{1}{aH}$
- Horizon crossing: $k = aH$



Inflationary Fluctuations

- The action:

$$S = \int d^4x \left[\frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

gravity inflaton field

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- Expand into background and fluctuations:

$$S \simeq S_0 + S_2$$

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gravity inflaton field

- Expand into background and fluctuations:

$$S \simeq S_0 + S_2$$

- With a few re-definitions, looks like a free field with time-dependent mass:

$$S_2 = \frac{1}{2} \int d^3x d\eta \left[\left(\frac{dv}{d\eta} \right)^2 - (\nabla v)^2 + \frac{d^2 z}{d\eta^2} \frac{v^2}{z} \right]$$

$$v \equiv z M_p \mathcal{R} \qquad z \equiv \frac{a^2}{H^2} \left(\frac{d\phi}{dt} \right)^2$$

Inflationary Fluctuations

- Quantize v, choose the Bunch-Davies vacuum, and find the correlation functions:

$$P(k) = Ak^{n_s - 1}$$

$$A = \frac{V^3}{12\pi^2 (\partial_\phi V)^2 M_P^6} \qquad n_s - 1 = 2\eta - 6\epsilon$$

$$n_s < 1 \qquad n_s > 1$$

Red

Blue

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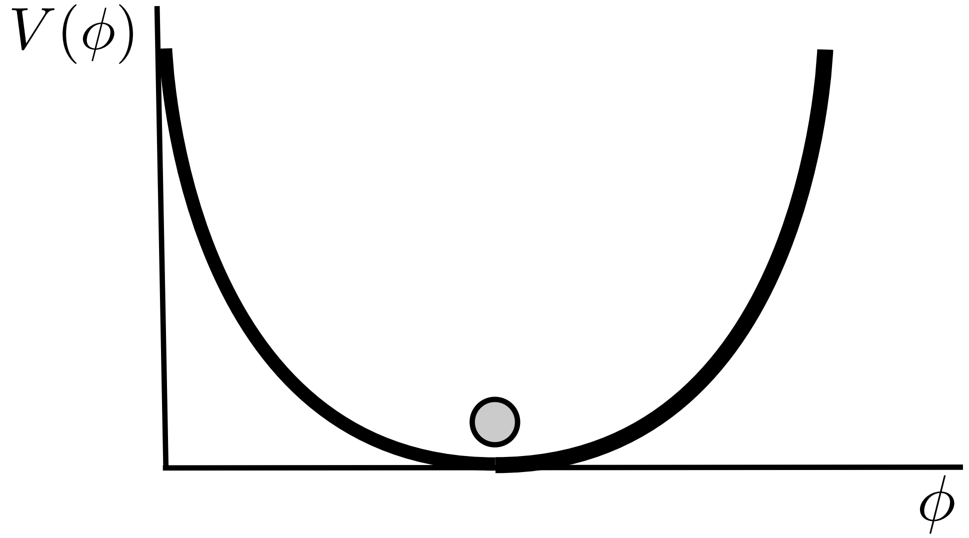
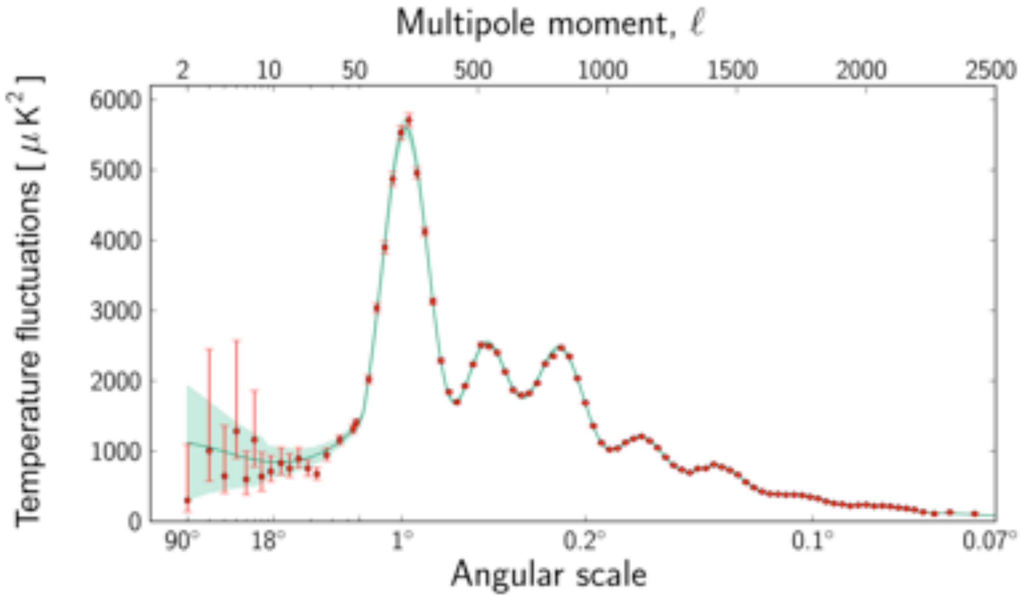
Red Blue

- Single-field slow-roll inflation: small non-gaussianity (to have slow-roll, interactions must be small).

Inflationary Fluctuations



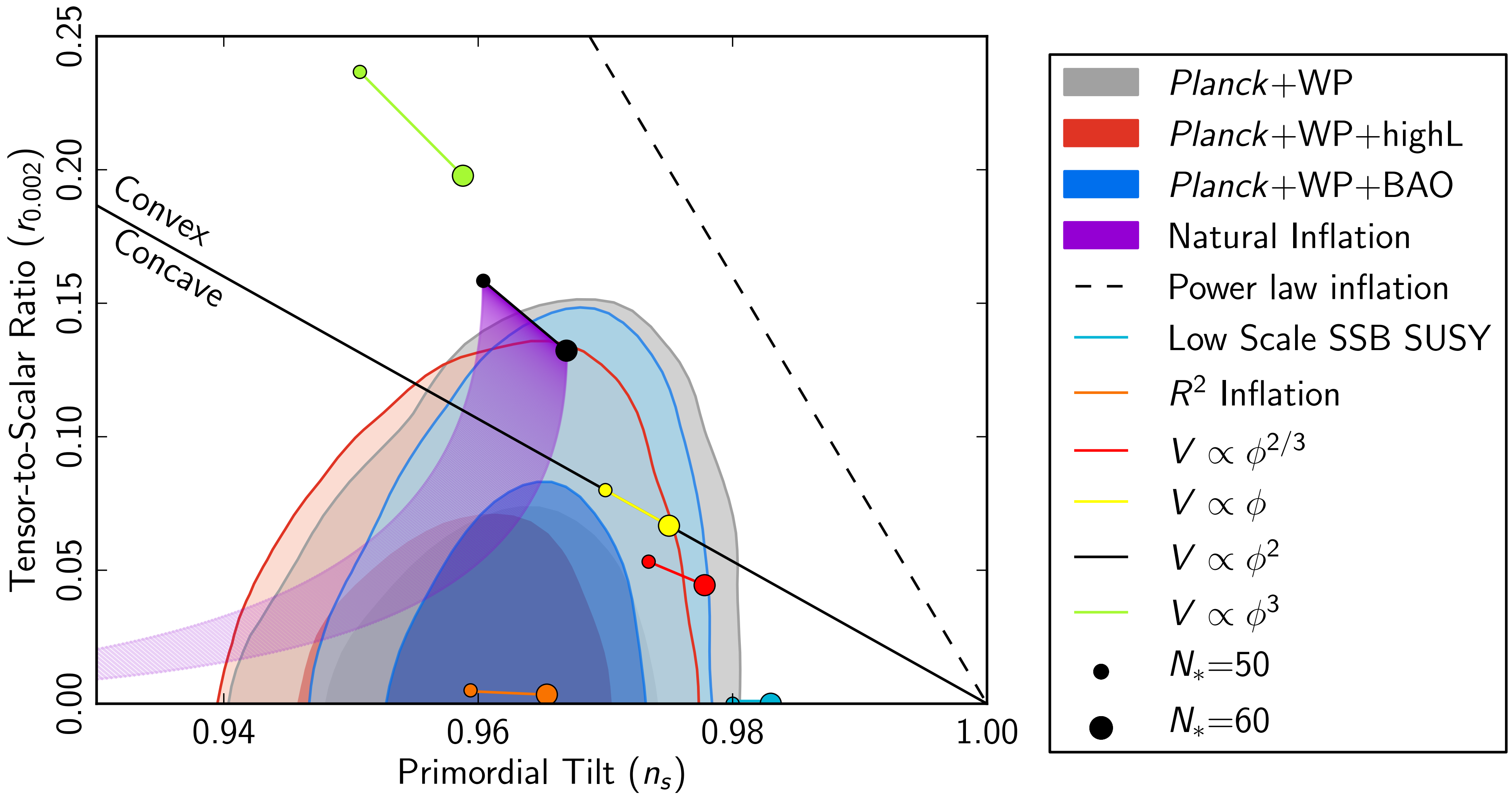
$$P(k)$$



Inflationary Fluctuations

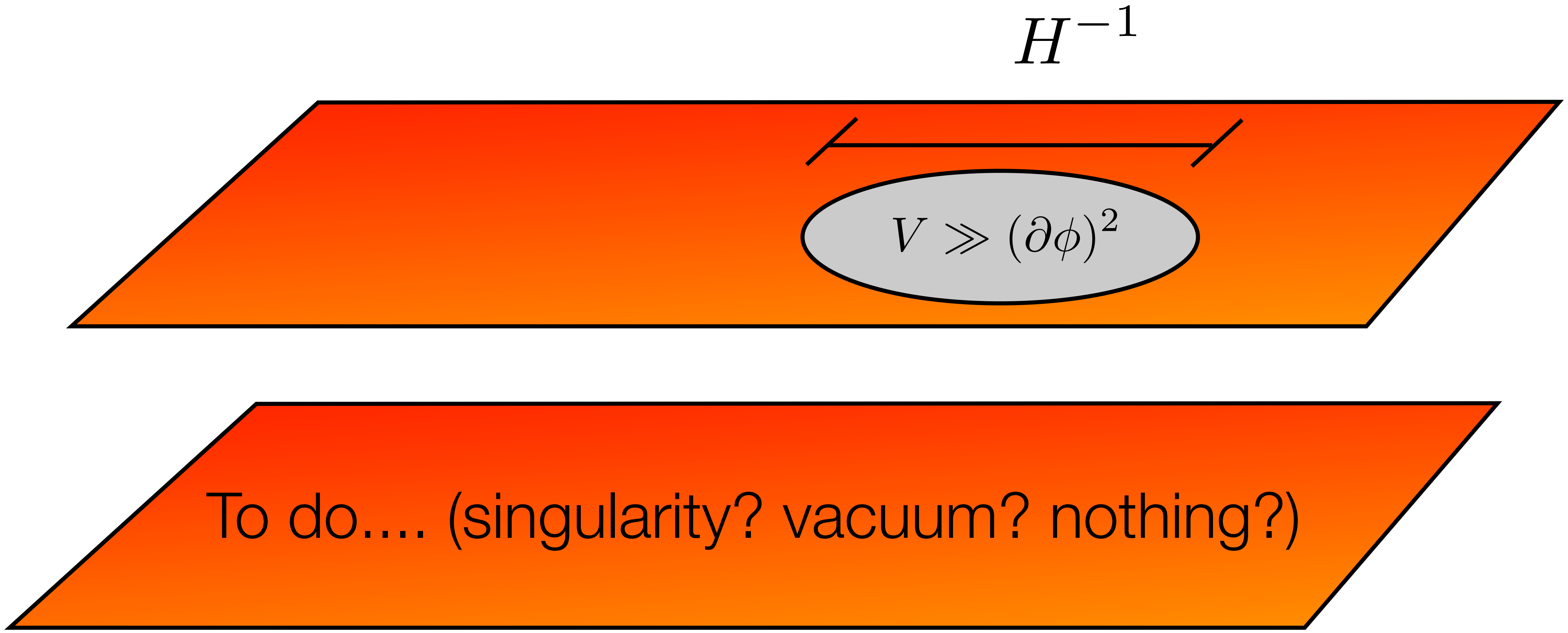
- Tensor modes:

$$A_T = \left(\frac{H}{2\pi}\right)^2 \qquad r \equiv \frac{A_T}{A} = 16\epsilon$$



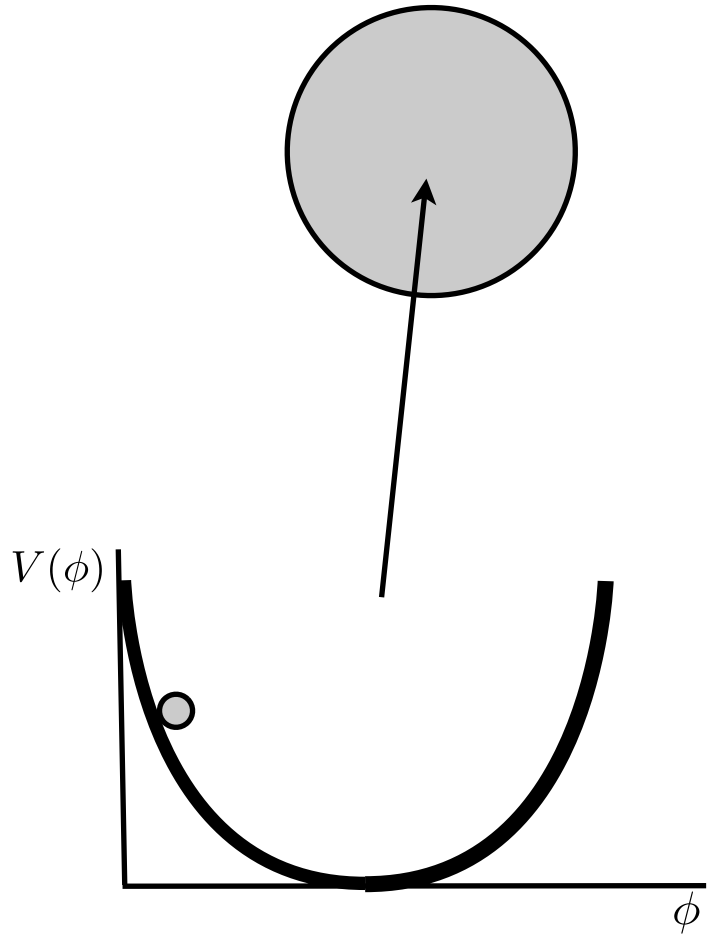
Initial Conditions for Inflation

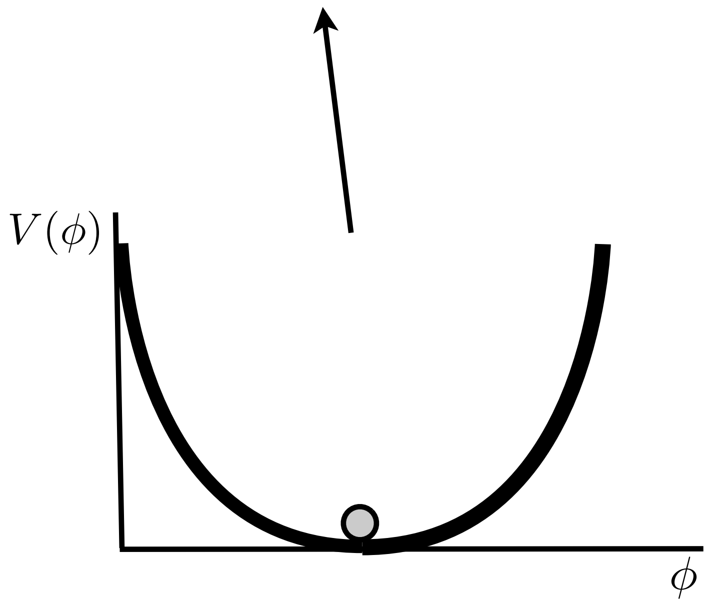
- Inflation, once it gets off the ground, can predict everything we observe about the linear universe.
- Under what conditions can inflation begin?



Inflation in the Lab

- What happens if I try to make inflation happen in the lab?





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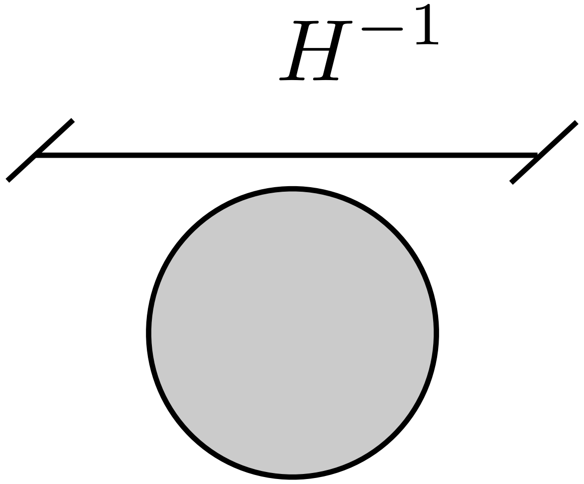

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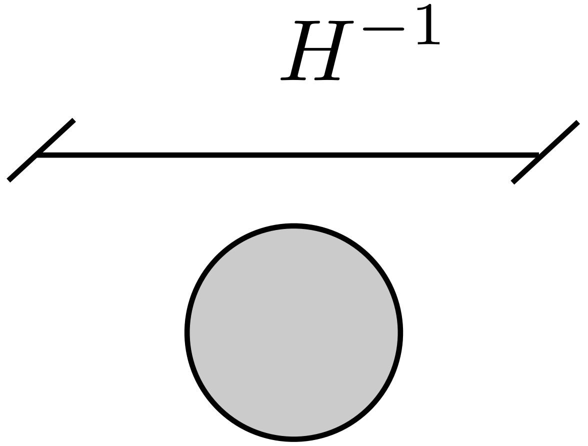

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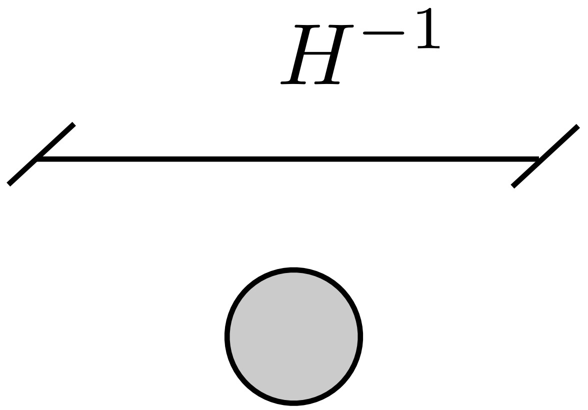

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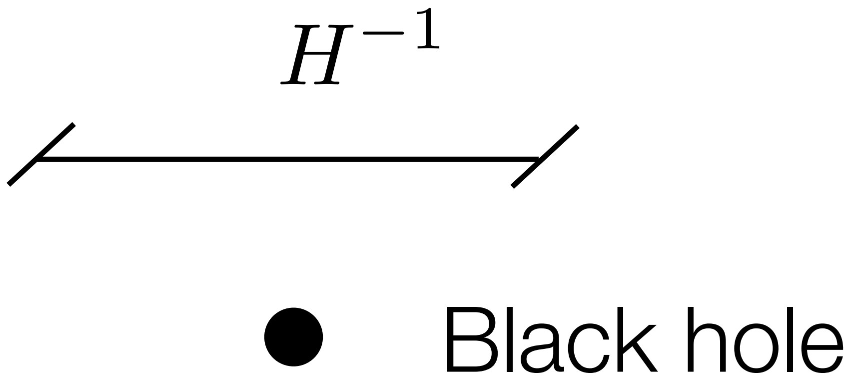

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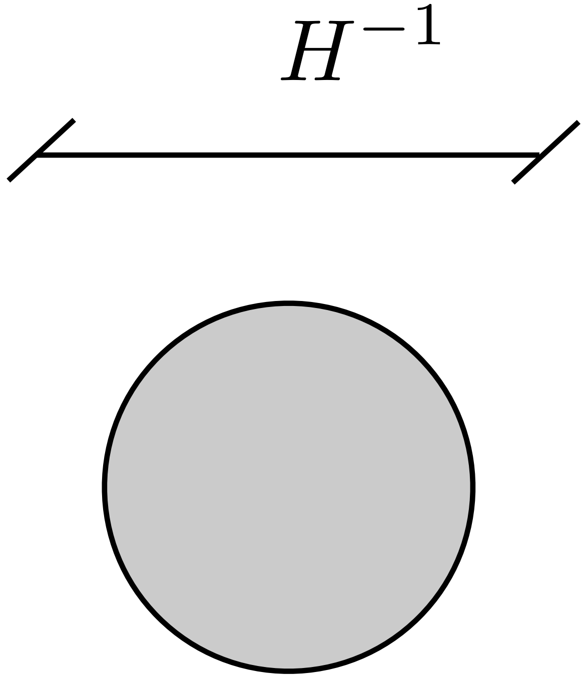

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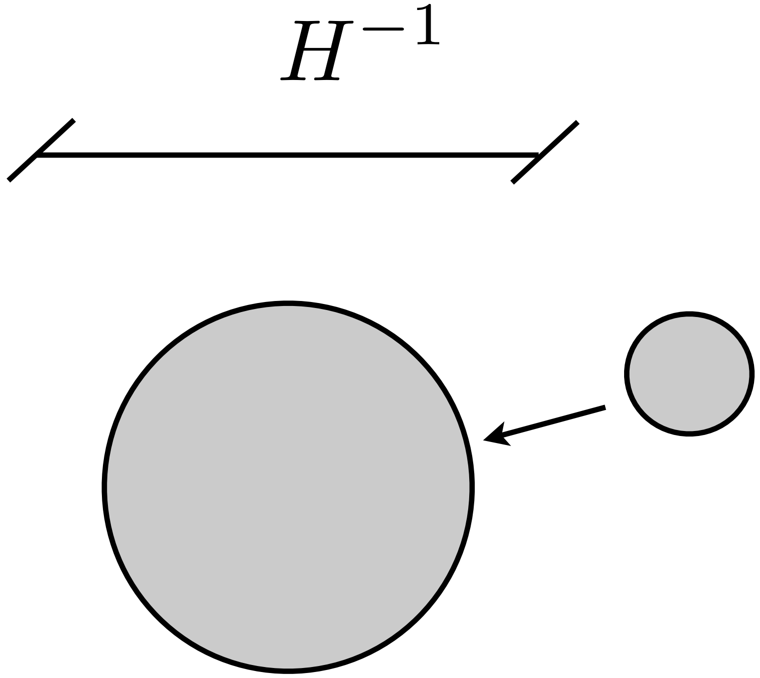

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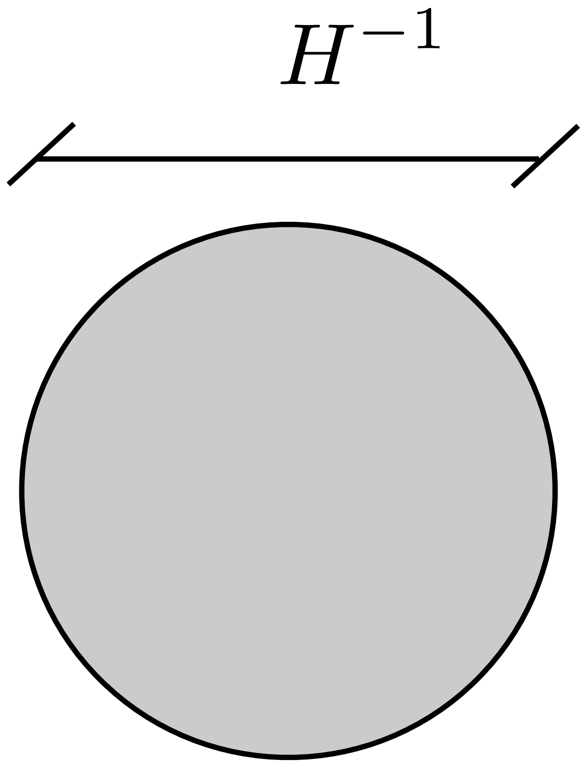

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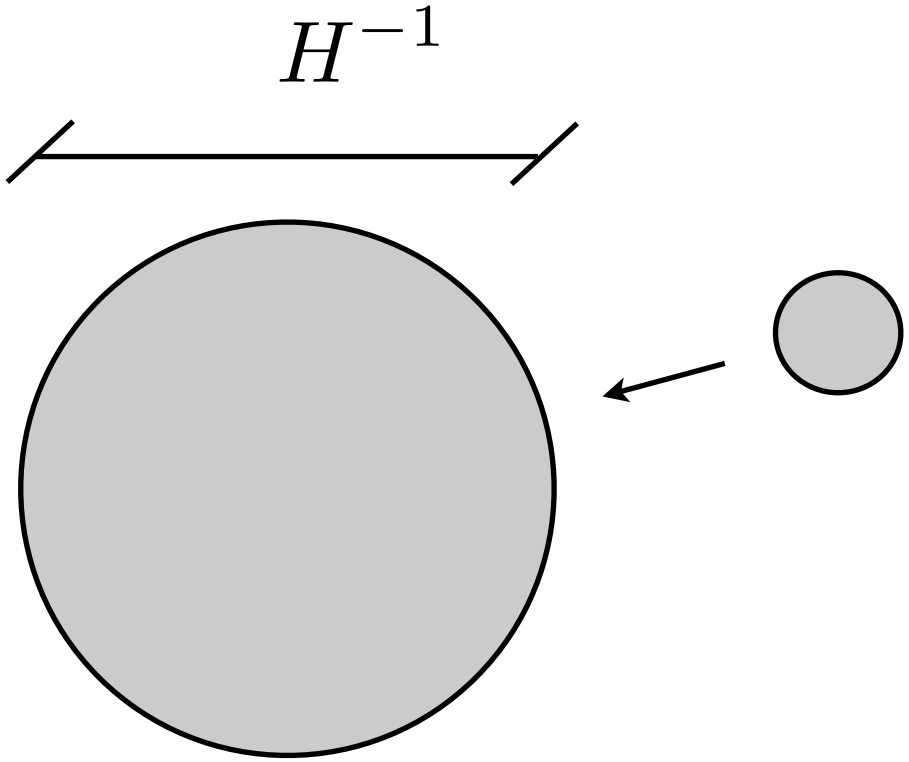

Inflation in the Lab

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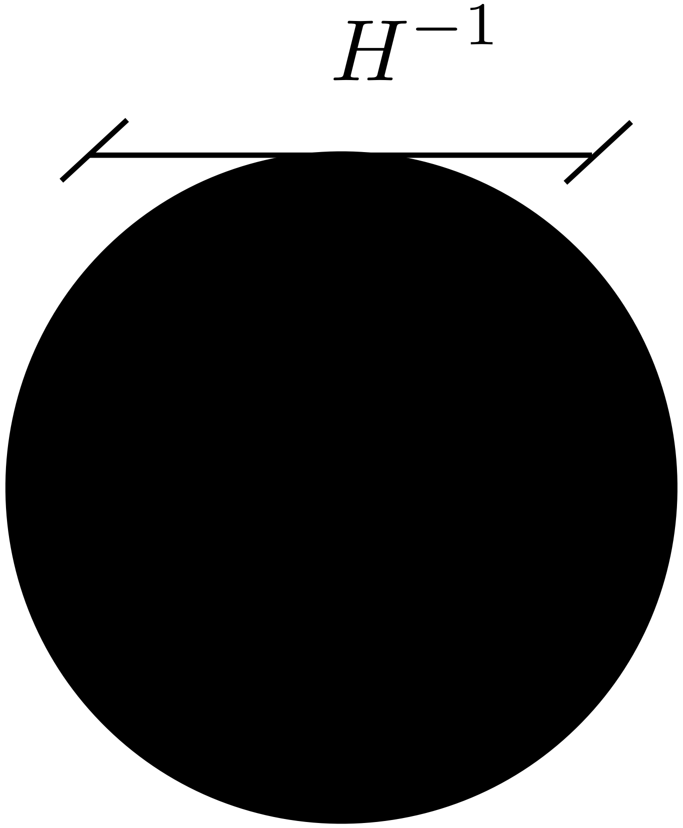
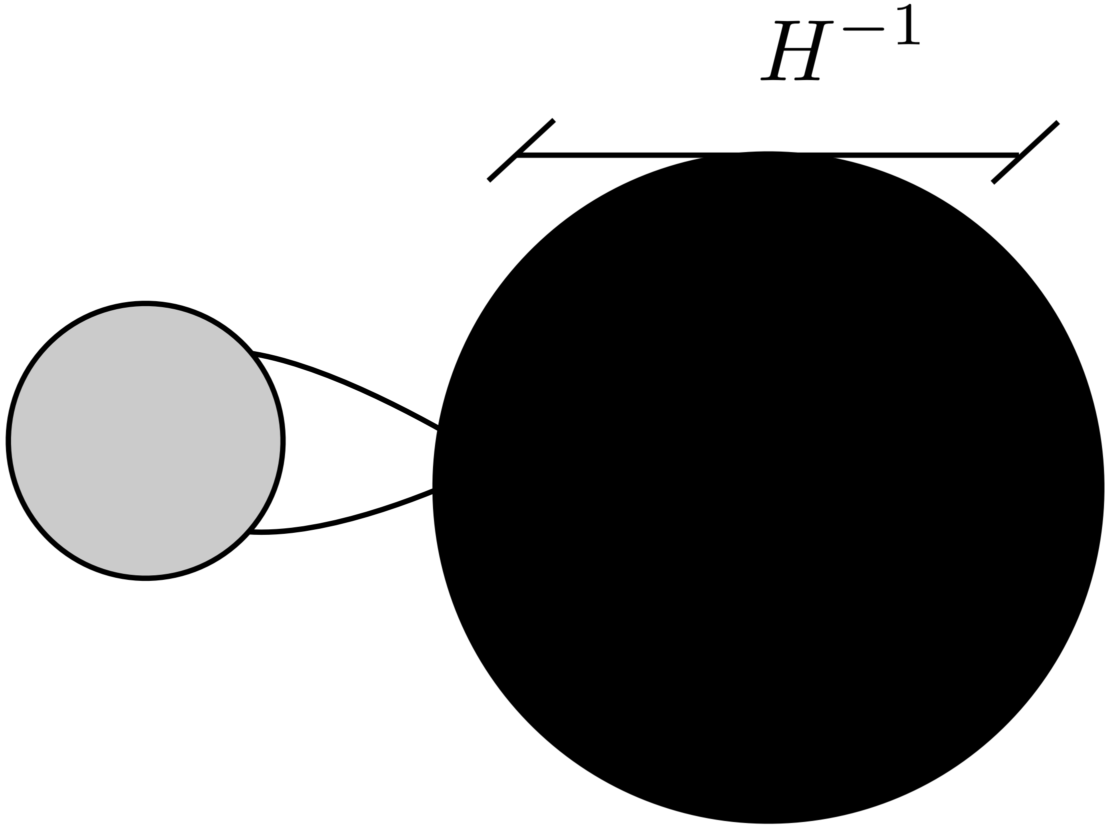


Black hole

Classically, you cannot find a way to make an inflating region in the lab!

Inflation in the Lab

- What happens if I try to make inflation happen in the lab?

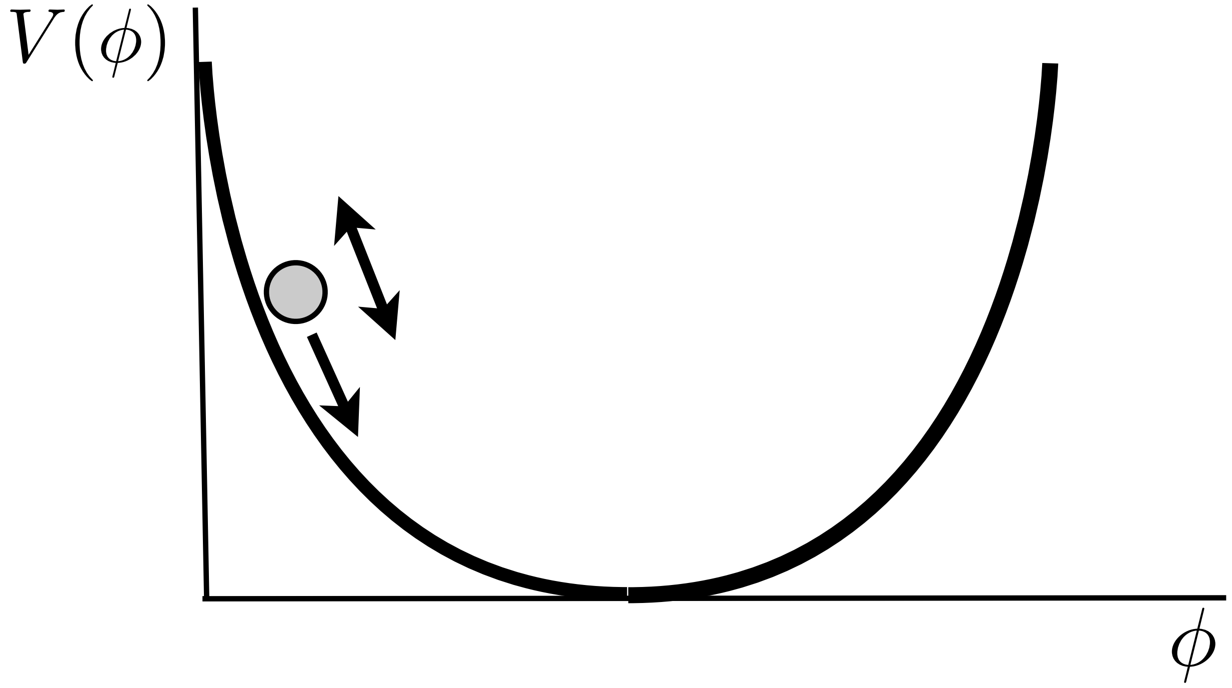


Black hole

Quantum mechanically, there is some probability that you succeed, but you will never know.

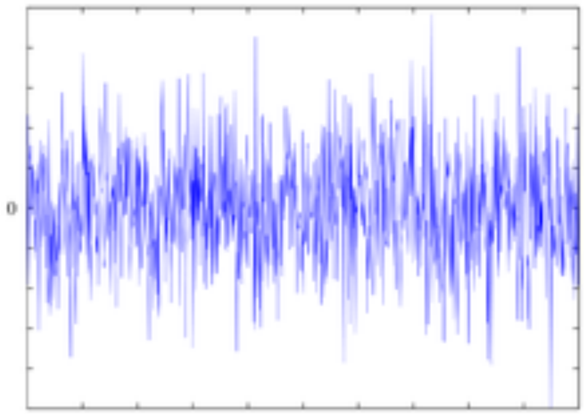
Eternal Inflation

- Stochastic eternal inflation:



$$\dot{\phi} = \frac{1}{3H}\left[-\frac{dV}{d\phi} + \frac{H^3}{2\pi}n(t)\right]$$

$$H^2 = \frac{8\pi G_N}{3}V(\phi)$$



$$\langle n(t)\rangle = 0 \qquad \langle n(t)n(t')\rangle = \delta(t-t')$$

- The jitters dominate the motion when: $\frac{dV}{d\phi} < H^3$

Eternal Inflation

- When this occurs, inflation becomes eternal.