Analytic Models of Chemical Evolution

- Pagel: *Nucleosynthesis and Chemical Evolution of Galaxies*, chs. 7-8
- Equations of GCE
- Specific models
  - closed box
  - leaky box
  - inflow model
Helpful references

• Beatrice Tinsley 1980, Fundamentals of Cosmic Physics, 5, 287
  – “Evolution of the Stars and Gas in Galaxies”

  – “Chemical Evolution of the Galaxy”

• Francesca Matteucci 2012, Springer
  – Chemical Evolution of Galaxies

  – “Alpha Element Distributions in Milky Way Satellite Galaxies”
Initial mass function

\[ \phi(m) \propto \frac{dN}{dm} \quad [M_\odot^{-1}] \]

\[ \int_{m_L}^{m_U} m \phi(m) \, dm = 1 \]

\[ \phi(m) = 0.17m^{-2.35} \quad \text{(Salpeter IMF)} \]

\[ \phi(m) = 13m^{-0.3} \quad \text{if } 0.01 \leq m < 0.08 \]

\[ \phi(m) = 0.29m^{-1.8} \quad \text{if } 0.08 \leq m < 0.5 \quad \text{(Kroupa 2001 IMF)} \]

\[ \phi(m) = 0.15m^{-2.7} \quad \text{if } 0.5 \leq m < 1.0 \]

\[ \phi(m) = 0.15m^{-2.3} \quad \text{if } m \geq 1.0 \]
Stellar lifetimes

$Z = 0.02 = Z_\odot$:

$$\tau = 950m^{-2.8} + 36m^{-0.65} + 1.5 \text{ Myr}$$

$Z = 0.001 = 10^{-1.3} Z_\odot$:

$$\tau = 650m^{-2.7} + 10m^{-0.90} + 1.7 \text{ Myr}$$
Basic GCE equations

\[ M = g + s \] 
\[ \frac{dM}{dt} = F - E \] 
\[ \frac{dg}{dt} = F - E + e - \psi \] 
\[ \frac{ds}{dt} = \psi - e \]

\[ e(t) = \int_{m_{\tau=t}}^{m_U} (m - m_{\text{rem}}) \psi(t - \tau(m)) \phi(m) \, dm \] 
\[ \frac{d}{dt} (gZ) = e_Z - Z \psi + Z_F F - Z_E E \]

\[ e_Z(t) = \int_{m_{\tau=t}}^{m_U} \left[ (m - m_{\text{rem}}) Z(t - \tau(m)) + m q_Z(m) \right] \psi(t - \tau(m)) \phi(m) \, dm \]
Basic GCE equations

\[ R = \int_{m_\tau}^{m_U} (m - m_{\text{rem}}) \phi(m) \, dm \]

\[ \alpha = 1 - R \]

\[ m_{\text{rem}} = \begin{cases} 
0.11m + 0.45 & \text{if } m \leq 6.8 \\
1.5 & \text{if } m > 6.8 
\end{cases} \]

\[ p_i = \alpha^{-1} \int_{m_\tau}^{m_U} m q_i(m) \phi(m) \, dm \quad [M_\odot] \]
Instantaneous recycling approximation

all stars born by time $t$, not accounting for mass loss

\[ S(t) = \int_0^t \psi(t') \, dt' \quad [M_\odot] \]

accounting for mass loss

\[ s(t) = \alpha S(t) \]

\[ \frac{dg}{dt} = F - E - \frac{ds}{dt} \]

\[ \frac{dg}{ds} = \frac{F - E}{\alpha \psi} - 1 \]

\[ \left( \frac{ds}{dt} = \alpha \frac{dS}{dt} = \alpha \psi \right) \]
Instantaneous recycling approximation

\[
e_Z(t) = \int_{m_{\tau=\tau}}^{m_U} [(m - m_{\text{rem}}) Z(t - \tau(m)) + mq_Z(m)] \psi(t - \tau(m)) \phi(m) \, dm
\]
\[
= \int_{m_i}^{m_U} [(m - m_{\text{rem}}) Z(t) + mq_Z(m)] \psi(t) \phi(m) \, dm
\]
\[
= \psi(RZ + \alpha \rho)
\]

\[
\frac{d}{dt}(gZ) = e_Z - Z \psi + Z_F F - Z_E E
\]
\[
\frac{d}{dS}(gZ) = \alpha \rho + RZ - Z + Z_F \frac{F}{\psi} - Z_E \frac{E}{\psi}
\]

\[
\frac{d}{ds}(gZ) = \rho - Z \left(1 + \frac{E}{\alpha \psi}\right) + Z_F \frac{F}{\alpha \psi}
\]

(from slide 5)
Instantaneous recycling approximation

\[
I = Z \frac{dg}{ds} + g \frac{dZ}{ds}
\]

\[
g \frac{dZ}{ds} = p + (Z_F - Z) \frac{F}{\alpha\psi}
\]

\[
\frac{dg}{ds} = \frac{F - E}{\alpha\psi} - 1 \quad \text{(from slide 7)}
\]

\[
\frac{d}{ds} (gZ) = p - Z \left(1 + \frac{E}{\alpha\psi}\right) + Z_F \frac{F}{\alpha\psi}
\]
Instantaneous recycling approximation

\[ M(t) = s(t) + g(t) = M_0 - M_{ej} + M_{accr} \]

\[ \frac{d}{ds} (gZ) = p - Z \left( 1 + \frac{dM_{ej}}{ds} \right) + Z_F \frac{dM_{accr}}{ds} \]

\[ g \frac{dZ}{ds} = p + (Z_F - Z) \frac{dM_{accr}}{ds} \]
Closed box

Posit that:
\[ g(t) + s(t) = M = \text{constant} \]
\[ Z = Z(t) \]
\[ g(0) = M \]
\[ Z(0) = S(0) = 0 \]

not a function of position; instantaneous mixing
Closed box + IRA

\[ dg = -ds \]

\[ g \frac{dZ}{ds} = p + (Z_F - Z) \frac{dM_{\text{accr}}}{ds} \]  

(from slide 10)

\[ g \frac{dZ}{ds} = -g \frac{dZ}{dg} = p \]

\[ Z = -p \int_0^t \frac{dg}{g} \]

\[ = -p \left[ \ln g(t) - \ln g(0) \right] \]

current metallicity of gas

No dependence on SFR (ψ)!

\[ Z = p \ln \frac{M}{g} \]

\[ = p \ln \mu^{-1} \]

\[ \mu = \frac{g}{M} \]  
gas fraction
Closed box + IRA

\[ z \equiv \frac{Z}{p} \]

\[ \langle z \rangle = \frac{1}{s} \int_{0}^{s} z(s') \, ds' \]

\[ = \frac{1}{s} \int_{0}^{s} \ln \left( \frac{s' + g}{g} \right) \, ds' \]

\[ \mu \equiv \frac{g/M}{M-s} = 1 - \frac{s}{M} \]

\[ = \frac{1}{1-\mu} \int_{1}^{\mu} \ln \mu' \, d\mu' \]

\[ \int \ln x \, dx = x \ln x - x \]

\[ = \frac{(\mu \ln \mu - \mu) + 1}{1-\mu} \]

\[ = 1 + \frac{\mu \ln \mu}{1-\mu} \]

average metallicity of stars
Closed box + IRA

\[ z = -\ln \mu \]
\[ = -\ln \frac{g}{M} \]

\[ \frac{g}{M} = 1 - \frac{s}{M} \]
\[ = e^{-z} \]

\[ \frac{s(z)}{M} = 1 - e^{-z} \]

\[ z \frac{ds}{dz} \propto \frac{ds}{d \log z} \propto z e^{-z} \]

MDF

No dependence on SFR (\( \psi \))!
Closed box + IRA
Closed box + IRA

\[
< Z > = p \left[ 1 + \frac{\mu \ln \mu}{1 - \mu} \right]
\]

\[< Z > \leq p \quad \text{always}\]

\[
\lim_{\mu \to 0} < Z > = p
\]

Simple Model = Closed Box

Graph showing the comparison between Canes Venatici I and the Simple Model.
Leaky box + IRA

\[ \frac{dg}{ds} = -\frac{E}{\alpha \psi} - 1 \equiv -\eta - 1 \]

constant \hspace{1cm} \text{outflow tracks SFR}

\[ g \frac{dz}{ds} = 1 \]

\[ g \frac{dz}{dg} = -\frac{1}{1+\eta} \]

\[ \frac{g}{M_0} = e^{-(1+\eta)z} \]

\[ \frac{ds}{d \log z} \propto z e^{-(1+\eta)z} \]

\[ p \rightarrow \frac{p}{1 + \eta} \]
Fig. 8.19. Distribution function of oxygen abundances of 132 G-dwarfs in the solar cylinder, binned in intervals of 0.1 in [O/H]. Triangles show the data points after Pagel (1989ab), based on a reanalysis of those discussed by Pagel & Patchett (1975), and boxes show lower and upper limits based on a new discussion of the dependence of the scale height on age and metallicity by Sommer-Larsen (1991a). The dotted curve shows predictions of an instantaneous Simple model with an initial enrichment [O/H] = -1.1 from the halo. The other model curves are discussed below. After B.E.J. Pagel, 'Abundances in Galaxies', in H. Oberhummer (ed.), *Nuclei in the Cosmos*, p. 98, Fig. 9. ©Springer-Verlag Berlin Heidelberg 1991.
Extreme inflow model

- gas inflow keeps gas mass constant
  \[ g = \text{constant} \]
  \[ Z_F = 0 \]
  \[ E = 0 \]
- inflowing gas is metal-free
- no outflow

\[
\frac{dg}{dt} = F - E - \frac{ds}{dt} = 0
\]
\[ F = \frac{ds}{dt} = \alpha \psi \]  
(from slide 7)

\[
\frac{g}{dZ} \frac{dZ}{ds} + \frac{Z}{g} = p + (Z_F - Z) \frac{F}{\alpha \psi}
\]
\[ Z \]  
(from slide 9)

\[
Z = Z_0 e^{-s/g} + p(1 - e^{-s/g})
\]

\[
\lim_{s/g \to \infty} Z = p
\]
“Best accretion model” (Lynden-Bell 1975)

Posit that:

\[ E = Z_F = 0 \]
\[ g(s) = (1 - s/M)(1 + s - s/M) \]
- simplest quadratic possible
- has a maximum in \( g \)
- smooth decay in \( g \) as star formation ebbs

if \( M = 1, \ g(s) = 1 - s \) — closed box

\[ \frac{d}{ds} (gZ) = p - Z \quad \text{(from slide 8)} \]

\[ z(s) = \left( \frac{M}{1 + z - s/M} \right)^2 \left[ \ln \frac{1}{1 - s/M} - \frac{s}{M} \left(1 - \frac{1}{M}\right) \right] \]

\[ \frac{ds}{d \ln z} = \frac{z[1 + s(1 - 1/M)]}{(1 - s/M)^{-1} - 2z(1 - 1/M)} \]
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Fornax dwarf spheroidal galaxy
Numerical models