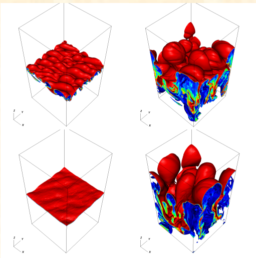
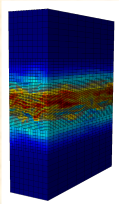
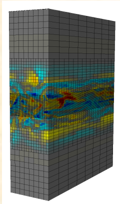
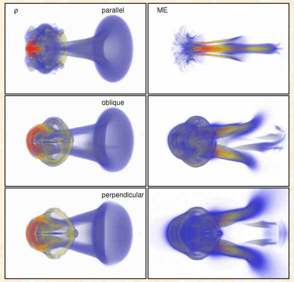
All about Athena



(Five lectures)







Jim Stone
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Outline of lectures

Lecture 1. Introduction to basic algorithm

Lecture 2. Grids in grid codes

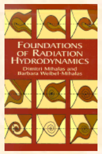
Lecture 3. Extra physics

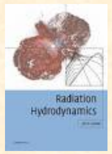
Lecture 4. Radiation hydrodynamics

Lecture 5. Example applications; future developments

Outline of Lecture 4:

1. Radiation hydrodynamics.
2. Numerical methods for radiation hydrodynamics.
   - Full transport methods
   - Flux-limited diffusion
3. Radiation hydrodynamics in Godunov schemes.
4. Future of grid-based methods.





Foundations of Radiation Hydrodynamics

Numerical MHD is easy compared to radiation hydrodynamics.

Some of the reasons why radiation MHD is hard:
- Which equations (transfer equation or its moments)?
- Which frame (co-moving, mixed-frame, fully relativistic)?
- Proper closure of moment equations.
- Mathematical problem changes in different regimes: *hyperbolic* in streaming limit, mixed *hyperbolic-parabolic* in diffusion limit.
- Wide range of timescales requires semi-implicit methods.
- Frequency dependence adds another dimension to solution
- Non-LTE effects requires modeling level populations.

This complexity means that radiation hydrodynamics means different things to different people.

In some cases, only need to include energy transport via material-radiation energy exchange term:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho v] = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + P] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+P)\mathbf{v}] = -g^0$$

$$g^0 = \int d\nu \int_{4\pi} d\Omega (j_\nu - \kappa_\nu k_\nu)$$

Optically thin cooling.
Heating by (ionizing) radiation.

Examples: diffuse ISM, HII regions.

In some cases, may need to include energy transport by diffusion (in optically thick regions) as well as material-radiation energy exchange term:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho v] = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + P] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+P)\mathbf{v}] = -g^0$$

$$\frac{\partial E_r}{\partial t} + \nabla \cdot D \nabla E_r = g^0$$

$$g^0 = \int d\nu \int_{4\pi} d\Omega (j_\nu - \kappa_\nu k_\nu)$$

Examples: dense ISM, protostellar disks

In some cases, may "only" need to include momentum exchange terms.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho v] = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + P] = -\mathbf{g}$$

$$\mathbf{g} = \frac{1}{c} \int d\nu \int_{4\pi} d\Omega \mathbf{n} (j_\nu - \kappa_\nu k_\nu)$$

e.g. line-driven winds (assuming gas is isothermal).

Of course, computing **g** can be extremely difficult!

In some cases, need to include *both* energy *and* momentum exchange terms.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho v] = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + P] = -\mathbf{g}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+P)\mathbf{v}] = -g^0$$

$$\frac{\partial E_r}{\partial t} + \nabla \cdot \mathbf{F}_r = g^0$$

$$\frac{1}{c^2}\frac{\partial \mathbf{F}_r}{\partial t} + \nabla \cdot \mathsf{P}_r = \mathbf{g}$$

$$g^0 = \int d\nu \int_{4\pi} d\Omega (j_\nu - \kappa_\nu k_\nu)$$

$$\mathbf{g} = \frac{1}{c} \int d\nu \int_{4\pi} d\Omega \mathbf{n} (j_\nu - \kappa_\nu k_\nu)$$

Examples:
radiation-dominated disks
core-collapse SN

All of these problems could be called "radiation hydrodynamics".

Obviously, the numerical methods required in each regime are very different.

Transfer equation.

Fundamental description of the radiation field is the frequency-dependent transfer equation

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \nabla \cdot (\mathbf{n} I_\nu) = j_\nu - \kappa_\nu I_\nu$$

Can be thought of as the "collisionless Boltzmann equation for photons", so that *I* is the "photon distribution function".

Just like the fluid equations, can take moments over phase space (angles) and frequency to derive a set of moment equations.

Why? Reduces dimensions of problem, making it easier to solve.

Grid-based method versus particles for radiation transfer

Even though we use a grid for the MHD, we could still choose to use either a grid or particles (Monte Carlo) to solve the transfer equation.

Grid:

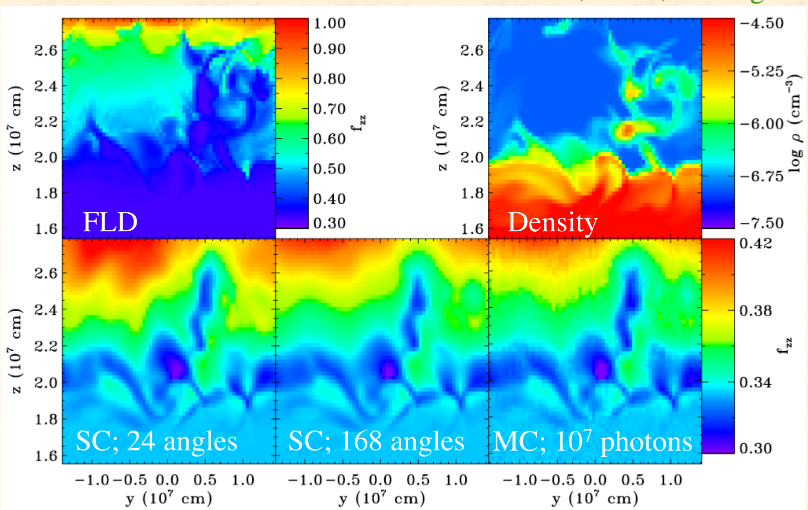
More accurate and less noise
Difficult to extend to include scattering, and line-transport
Very expensive

Particles (Monte Carlo):

Very flexible, easy to extend to frequency-dependent transport, etc.
Embarassingly parallel
Noisy, especially in optically thick regions

The noise in MC is a big problem.

Davis, Stone, & Jiang 2012



Eddington factor $f_z = P_{zz}/E$. *MC method was 100x slower!*

Ionizing radiation transport

Application: growth of HII regions in ISM. Solve MHD equations for 2-fluid (ions + neutrals) medium, including heating, cooling, photoionization, and recombination.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{v}) = 0,$$

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla\cdot(\rho \boldsymbol{v}\boldsymbol{v} - \boldsymbol{B}\boldsymbol{B}) + \nabla P^* = 0,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla\cdot(\boldsymbol{v}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{v}) = 0,$$

$$\frac{\partial E}{\partial t} + \nabla\cdot[(E + P^*)\boldsymbol{v} - \boldsymbol{B}(\boldsymbol{B}\cdot\boldsymbol{v})] = \mathcal{G} - \mathcal{L},$$

$$\frac{\partial \rho_n}{\partial t} + \nabla\cdot(\rho_n \boldsymbol{v}) = \mathcal{R} - \mathcal{I},$$

$$\mathcal{I}_{\rm ph} = \sigma \rho_n \sum_n \frac{s_n}{4\pi|\boldsymbol{x} - \boldsymbol{x}_n|^2} e^{-\tau(\boldsymbol{x},\boldsymbol{x}_n)}.$$

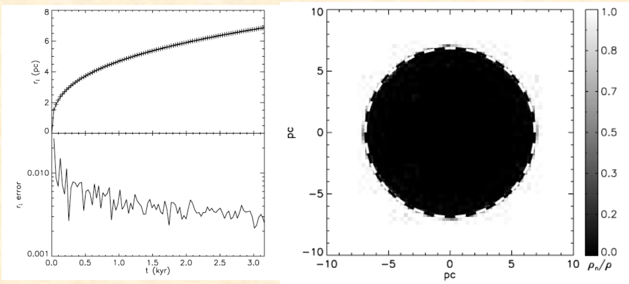
$$\tau(\boldsymbol{x},\boldsymbol{x}_n) \approx \int_{\boldsymbol{x}_n}^{\boldsymbol{x}} (\sigma n_{\rm H} + \sigma_d n)\, d\ell$$

Challenge: compute optical depth from every point source to every grid cell.

Algorithm

Krumholz, Stone, & Gardiner (2007)

- Use adaptive ray-tracing method of Abel & Wandelt (2002) and Whalen & Norman (2006) using HEALPix to compute ionization rate in each cell
- Limit cooling in mixed cells (last lecture!)
- Tests: propagation of R- and D-type I-fronts.



Test of growth of R-type front with no recombinations

Euler equations + Maxwell's equations + zeroth and first moment equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + P + B^2/2 - \mathbf{B}\mathbf{B}) = -\mathbb{P}\mathbf{S_M}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P)\mathbf{v} + (B^2/2)\mathbf{v} - \mathbf{B}(\mathbf{B}\cdot\mathbf{v})] = -\mathbb{P}\mathbb{C}S_E$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

$$\frac{\partial E_r}{\partial t} + \mathbb{C}\nabla \cdot \mathbf{F}_r = \mathbb{C}S_E$$

$$\frac{\partial \mathbf{F}_r}{\partial t} + \mathbb{C}\nabla \cdot \mathsf{P}_r = \mathbb{C}\mathbf{S_M}$$

E_r, F_r, P_r are radiation energy density, flux, pressure in Eulerian (fixed) frame.

Source terms are O(v/c) expansion of material-radiation interaction terms in fluid frame.

$$\mathbf{S_M} = -(\sigma_a + \sigma_s)\left(\mathbf{F}_r - \frac{\mathbf{v}E_r + \mathbf{v}\cdot\mathsf{P}_r}{\mathbb{C}}\right) + \frac{\mathbf{v}}{\mathbb{C}}(\sigma_a T^4 - \sigma_a E_r)$$

Lowrie et al 1999

$$S_E = (\sigma_a T^4 - \sigma_a E_r) + (\sigma_a - \sigma_s)\frac{\mathbf{v}}{\mathbb{C}} \cdot \left(\mathbf{F}_r - \frac{\mathbf{v}E_r + \mathbf{v}\cdot\mathsf{P}_r}{\mathbb{C}}\right)$$

Pros: reduced problem to system of 3+1 (3D + time) PDEs

Cons: PDES are of mixed type. Hyperbolic/parabolic in different limits

Large range of timescales associated with widely varying characteristics: v, C_s, c

Source terms can be very stiff (much bigger than flux divergence)

Need closure relation, e.g $\mathbf{P}_r = \mathbf{f}\, E_r$ But how to compute $\mathbf{f}$?

Solving the closure problem: Flux-limited diffusion

Adopt the diffusion approximation everywhere $\mathbf{F} = -D\nabla E$

Superluminal transport in optically thin regions, unless flux is limited: $D = \frac{c\lambda}{\chi}$

$\lambda=\lambda(E)$ is limiter

These reduces the RMHD equations to a two-temperature diffusion problem.

Turner & Stone 2001

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$$

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla P + \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B} + \frac{1}{c}\chi_F\mathbf{F}$$

$$\rho\frac{D}{Dt}\left(\frac{E}{\rho}\right) = -\nabla\cdot\mathbf{F} - \nabla\mathbf{v} : \mathsf{P} + 4\pi\kappa_P B - c\kappa_E E$$

$$\rho\frac{D}{Dt}\left(\frac{e+E}{\rho}\right) = -\nabla\mathbf{v} : \mathsf{P} - P\nabla\cdot\mathbf{v} - \nabla\cdot\mathbf{F}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})$$

Pros: easy to solve

Cons: lost information about direction of flux

magnitude of flux in optically thin regions is ad-hoc

no radiation inertia (superluminal wave speeds)

no radiation shear viscosity

Flux limiter

Arbitrary function of E. Most popular form is due to Levermore & Pomraning (1981)

$$\lambda(R) = \frac{2+R}{6+3R+R^2} \qquad R = |\nabla E|/E$$

Main purpose of limiter is to give correct flux in optically thin and thick limits

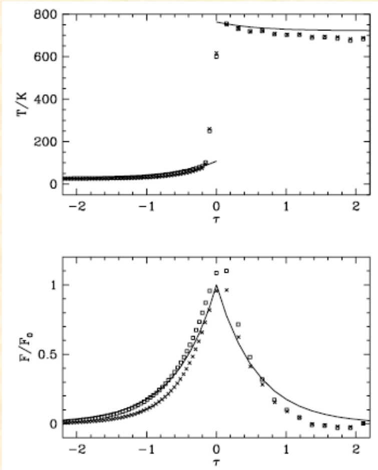
$$\lim_{R\to\infty}\lambda(R) = \frac{1}{R}$$ Optically thin limit, F ~ cE

$$\lim_{R\to 0}\lambda(R) = \frac{1}{3}$$ Optically thick limit, F ~ Grad(E)

Test of FLD: subcritical shock

X = Minerbo limiter

□ = Levermore & Pomraning limiter



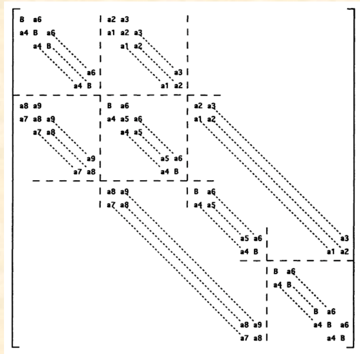
Solid line = approx analytic solution from Zel'dovich & Raizer (1967)

Parameters same as in Sincell, Gehmeyr & Mihalas (1999)

Implicit differencing.

Material-radiation interaction and radiation transport terms have a very restrictive time step limit, and must be solved implicitly.

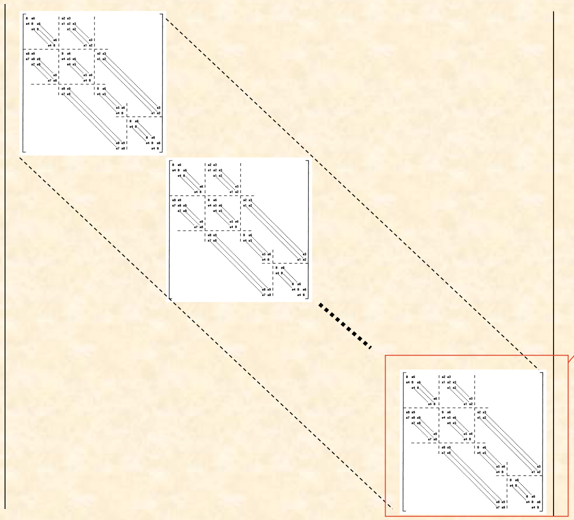
Equations are nonlinear in unknowns, so must use Newton-Raphson iteration. Requires solving large sparse-banded matrix for *every* NR iteration.



Matrix solved for each NR iteration is very sparse, so use iterative methods like GMRES or ICCG.

Schematic of matrix in 2D

In 3D, matrix to be solved in *each* NR step is N^3 x N^3 where N is number of grid points along each dimension.

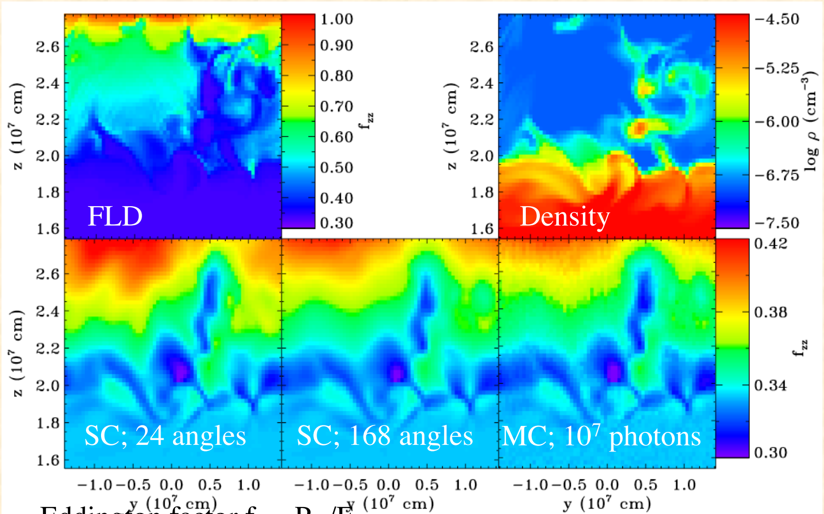


Equivalent to entire matrix in 2D problem

Reduced speed of light methods

- To avoid implicit differencing, simply assume c is slightly larger than sound speed
- Affects dynamics, so must be used with caution.
- E.g. Skinner & Ostriker (2013)

FLD doesn't give correct values even in optically thin regions



Eddington factor $f_z = P_{zz}/E$

Davis, Stone, & Jiang 2012

M1 closure

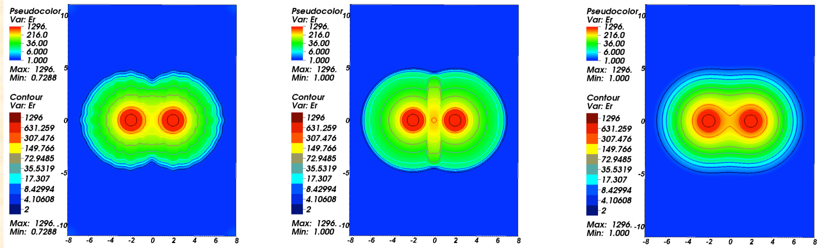
To avoid problems with FLD, new local closures have been tried

Most popular currently is M1 (Gonzalez et al 2007)

- Keeps flux as separate variable
- Uses local information to construct direction of flux

M1 fixes one problem (lack of shadows with FLD), but replaces it with another (photons collide and merge with M1)

Radiation energy density from two radiating spheres:



Formal soln. M1 FLD

Towards Better Numerical Methods

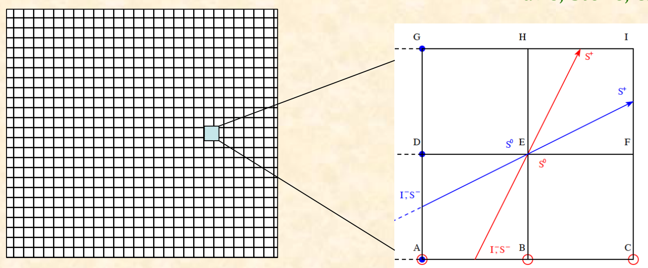
The three challenges to solving the coupled moment equations more generally:

- Need closure relation **P=fE**
  - Compute variable Eddington tensor (VET) from "snap-shot" solution of time-independent transfer equation.
- Source terms can be very stiff
  - Use modified Godunov method
- Wide range of timescales associated with v, C, c
  - Requires fully implicit (backward Euler) differencing of radiation moment equations

Variable Eddington Tensor

Compute using short characteristics to solve time-independent transfer equation along N_r rays per cell. $\partial I/\partial s = \kappa(S - I)$

Olson & Kunasz 1987;
Stone, Mihalas, & Norman 1992
Davis, Stone, & Jiang 2012



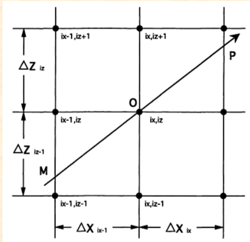
Method includes scattering and non-LTE effects using accelerated lambda iteration (ALI).

Trujillo Bueno & Fabiani Bendicho 1995

Then compute **f** directly from moments of I. $\mathsf{f} = \frac{\mathsf{P}_r}{E_r} = \frac{\int \hat{\mathbf{n}}\hat{\mathbf{n}} I d\omega}{\int I d\omega}$

Short versus long characteristics

Short characteristics (Kunasz & Auer 1988): solve along ray segments that cross a single zone, and interpolate I to start of next ray segment, $O(N^3)$ in 3D

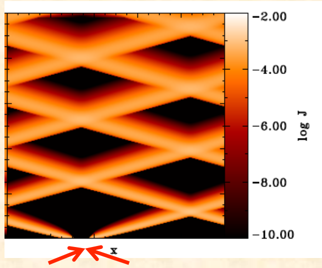


Long characteristics: for each cell, solve along rays that cross entire grid, $O(N^4)$ in 3D.

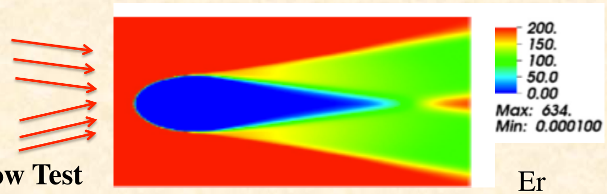
Short characteristics are much faster, but can have problems in treating point sources.

Tests of Transfer Solver

Davis, Stone, & Jiang 2012



Beam Test
Two "flashlight" beams in optically thin, periodic domain.



Er

Shadow Test
Optically thick, spherical cloud irradiated by two beams.

Modified Godunov method

Miniati & Colella 2007
Sekora & Stone 2010

Stable, 2nd order accurate scheme for handling stiff source terms.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + P + B^2/2 - \mathbf{B}\mathbf{B}) = -\mathbb{P}\mathbf{S_M}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P + B^2/2)\mathbf{v} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = -\mathbb{P}\mathbb{C}S_E$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

Uses modified wave speeds and eigenvectors to compute fluxes.

Semi-implicit (Picard iteration) scheme ensures stability.

Implicit solution of moment equations.

Method must be implicit to allow $\delta t > dx/c$.
Solving entire system of equations implicitly is expensive and inaccurate.

Instead, split fully-implicit solution of radiation moment equations from modified Godunov method for MHD equations.

$$\frac{\partial E_r}{\partial t} + \mathbb{C}\nabla \cdot \mathbf{F}_r = \mathbb{C}S_E$$

$$\frac{\partial \mathbf{F}_r}{\partial t} + \mathbb{C}\nabla \cdot \mathsf{P}_r = \mathbb{C}\mathbf{S_M}$$

Requires inverting large sparse matrix every time step. *This is usually the slowest step in the entire algorithm.*

Algorithm: lessons learned

- Even in strong conservative form, it is difficult to conserve energy exactly with iterative implicit solvers.
- Writing our own multigrid solver resulted in a faster algorithm than using canned libraries like *Hypre*, *PETSci*, or *LES*.
- Method is not really any slower than FLD, since matrix solve is about the same, and for LTE problems the RT solver is cheap.

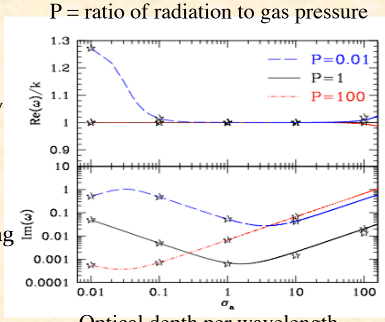
Test of Full Code: Linear Waves

Quantitative measure of error and convergence rate.

P = ratio of radiation to gas pressure

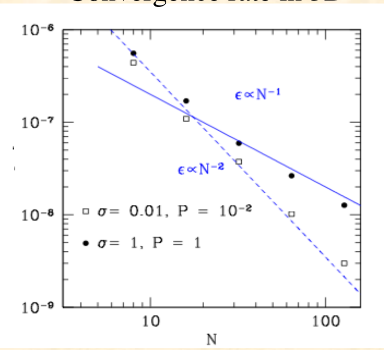
Phase Velocity

Damping Rate



Optical depth per wavelength

Convergence rate in 3D



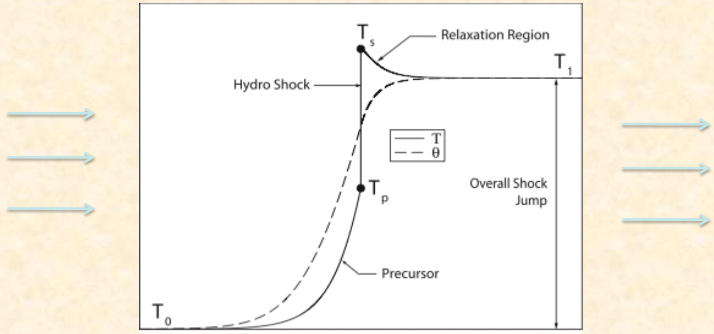
Stars are measured phase velocity and damping rate from 1D code.

Jiang et al. 2012

Radiation Shock Tests

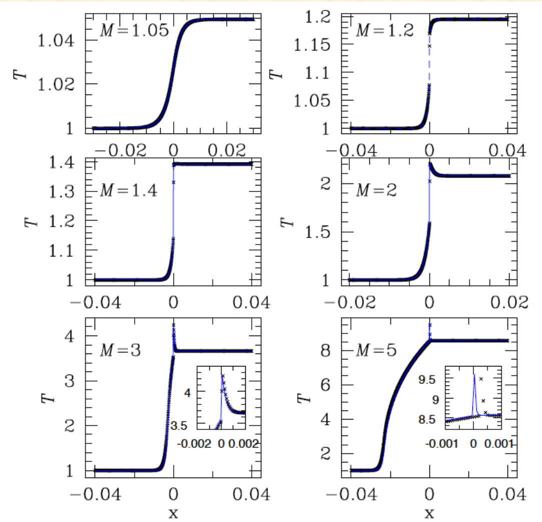
- 1D steady shock with pure absorption opacity
- Semi-analytic solution possible in nonequilibrium diffusion limit (Eddington approximation, f=1/3)

Lowrie & Edwards (2008)



Shock structure changes with different Mach numbers.

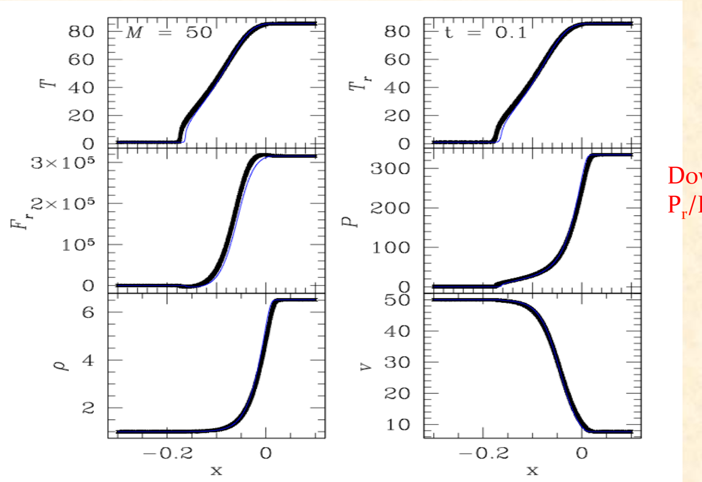
Structure of shock changes with Mach number



Solid = semi-analytic solution
Points = numerical solution

Jiang et al. 2012

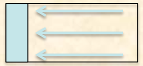
Radiation pressure dominated

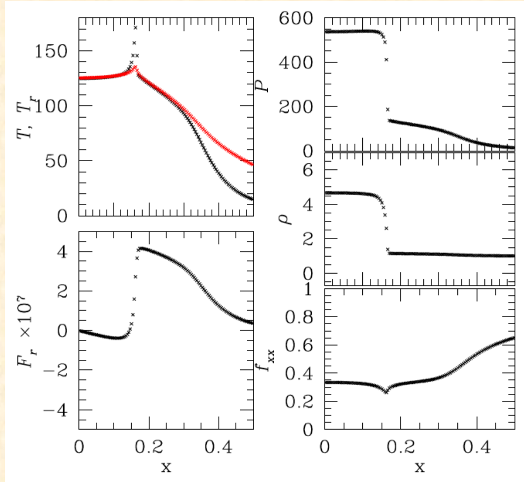


Downstream
$P_r/P \sim 10$

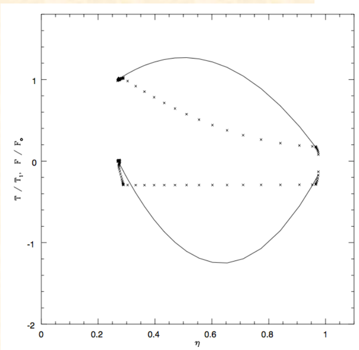
- Shock structure with full transport (VET)
- No semi-analytic solution known *(experiments?)*.

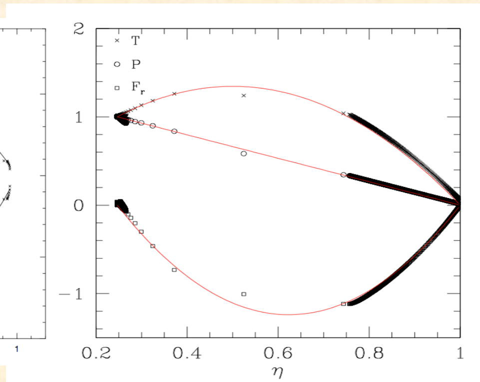
Shock formed by flow hitting reflecting wall





Radiation Shock with VET





(inverse compression ratio: ρ_0/ρ)

Sincell et al. (1996) using artificial viscosity.

Jiang et al. 2012

With Godunov scheme, profiles inside shock consistent with analytic estimate of Zeldovich & Raizer 1967

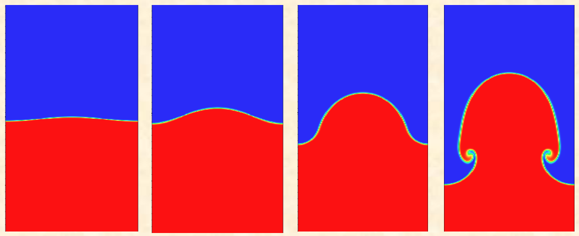
Test/Application: Rayleigh-Taylor instability.

Classic instability of heavy fluid accelerated by light fluid.
In hydrodynamics, growth rate $\gamma = (Agk)^{1/2}$ Atwood number $A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$

Goal of this work: how is instability modified when $P_{rad}/P_{gas} >> 1$, or when atmosphere is *supported* by radiation.

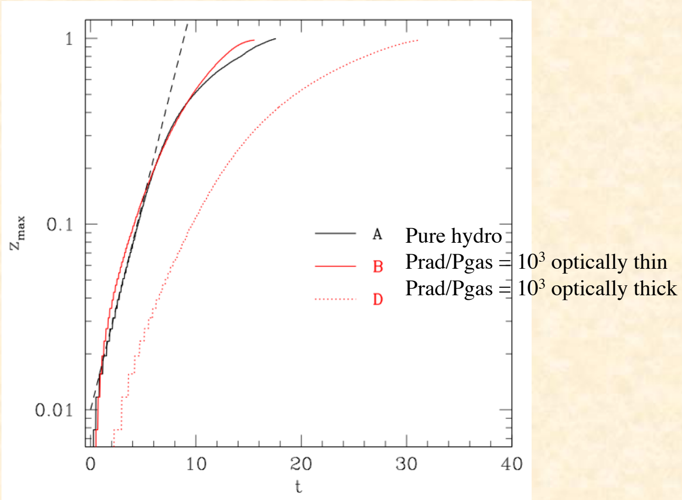
Jacquet & Krumholz 2011

Computational domain *L* x *2L*, 256 x 512 grid cells
Uniform vertical gravity *g*, density ratio = 4.
Scattering opacity *only*. (No equilibrium possible with non-zero absorption opacity)

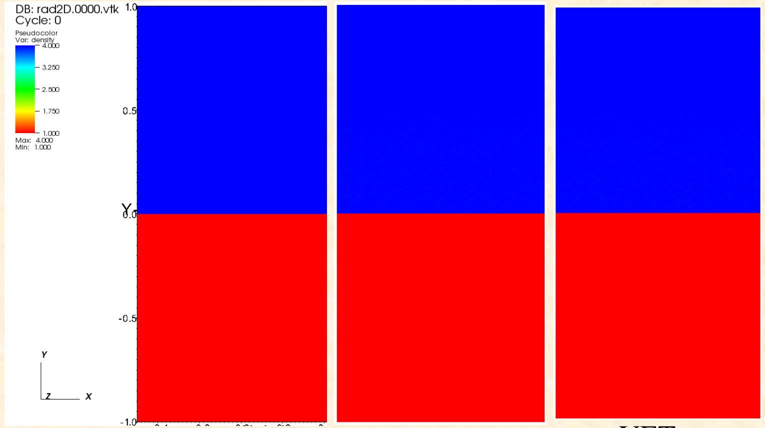


Evolution of single mode with Prad/Pgas = 10^3

Single mode growth rates.

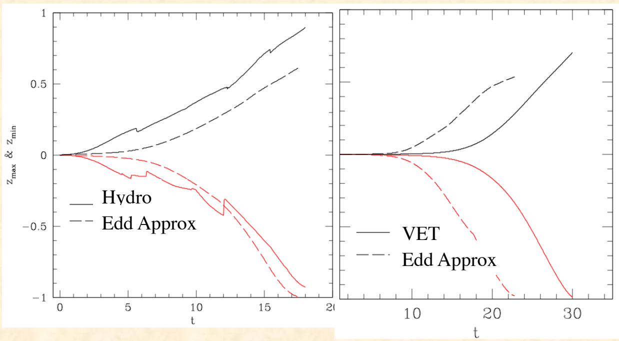


Multimode perturbations and mixing



hydrodynamics f=1/3 VET

Height of fingers and bubbles in nonlinear regime



Fingers fall faster than bubbles rise because radiation escapes from low density bubbles.
Growth is slower and less mixing with radiation.
Significant difference between diffusion approximation and VET.

Conclusions

- Radiation MHD is still an active area of research and methods are still being developed.
- Use full-transport methods that solve the transport equation directly. Do not use FLD or M1 closures.
- Directions for future:
  - Time-dependent transport for relativistic problems
  - Frequency-dependent transport