Finite *volume* methods for compressible MHD

Patrick Hennebelle
Thanks to
Romain Teyssier and Sébastien Fromang

Summary of the lecture

1) Introduction
   - stability
   - MHD equations, standard and conservative forms
   - Godunov-type methods and Riemann problems

2) Riemann solver
   - exact hydrodynamical solver
   - ROE solver
   - HLL type solvers

3) High order schemes

4) 1D and 2D Numerical tests: comparing the solvers

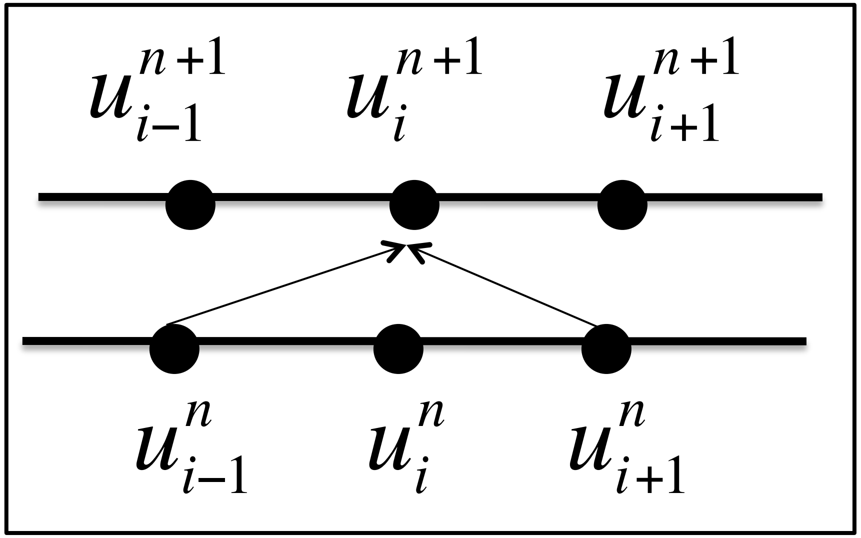
5) MultiD MHD
   - Specificity of the MultiD MHD equations
   - the methods

6) 2D Numerical tests: comparing the methods

Introduction

Explicit methods and stability

We consider the simple advection equation: $\partial_t u + a\partial_x u = 0$
a some constant.



Let us discretize it:

A possible and appealing choice is:

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

(subscript timestep, underscript position)

=>This method turns out to be unstable…

Physically this is because, information is “upwind”. It should come only from the regions from which the flow is coming. Can be interpreted as a negative viscosity otherwise.

Mathematically, this can be shown using von Neumann analysis.
Let Fourier transform the mesh:

$$u_i^n = \sum A_k^n \exp(-ikx_i)$$

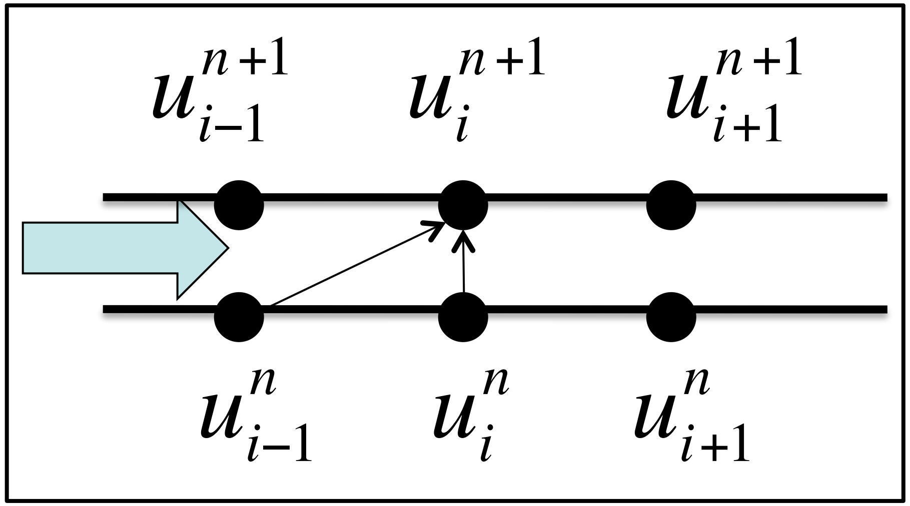
$$\Rightarrow A_k^{n+1} = A_k^n + \frac{a\Delta t}{2\Delta x}\left(-A_k^n \exp(-ik\Delta x) + A_k^n \exp(ik\Delta x)\right)$$

$$= A_k^n\left(1 + iC\sin(k\Delta x)\right), C = \frac{a\Delta t}{2\Delta x}$$

$$\Rightarrow \left|\frac{A_k^{n+1}}{A_k^n}\right|^2 = 1 + C^2\sin^2(k\Delta x) \geq 1$$

Thus, the modes are amplified at each time step leading to a strong instability.

Physically, the following discretization should get rid of this problem:



$$a > 0 \Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

$$a < 0 \Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_i^n}{\Delta x} = 0$$

Let us check this mathematically:

$$u_i^n = \sum A_k^n \exp(-ikx_i)$$

$$\Rightarrow A_k^{n+1} = A_k^n + \frac{a\Delta t}{\Delta x}\left(-A_k^n + A_k^n \exp(ik\Delta x)\right)$$

$$= A_k^n\left(1 + C(-1 + \exp(ik\Delta x))\right), C = \frac{a\Delta t}{2\Delta x}$$

$$\Rightarrow \left|\frac{A_k^{n+1}}{A_k^n}\right|^2 = 1 - 2C(1-C)(1-\cos(k\Delta x))$$

Thus, the scheme is stable as long as $C<1$ which is called the:
Courant condition.

Physical meaning is clear: information should come from the nearest upwind neighbours.

In the case of more complex equations entailing various wave propagation, information comes generally from both the left and the right neighbours, depending on the *wave* which is considered.

Important messages:

-discretization matters a lot

-information should be upwind

-time step is a crucial issue

MHD equations, standard and conservative forms

$$\partial_t \rho + \vec{V}.\vec{\nabla}\rho + \rho\vec{\nabla}.\vec{V} = 0$$

$$\rho(\partial_t \vec{V} + \vec{V}.\vec{\nabla}\vec{V}) = -\vec{\nabla}P + (\vec{\nabla}\times\vec{B})\times\vec{B}$$

$$= -\vec{\nabla}P - \vec{\nabla}\frac{\vec{B}^2}{2} + \vec{B}.\vec{\nabla}\vec{B}$$

$$\partial_t e + \vec{V}.\vec{\nabla}e + (\gamma-1)e\vec{\nabla}\vec{V} = 0$$

$$\partial_t \vec{B} + \vec{V}.\vec{\nabla}\vec{B} - \vec{B}.\vec{\nabla}\vec{V} + \vec{B}\vec{\nabla}.\vec{V} = 0$$

$$\vec{\nabla}.\vec{B} = 0$$

$$e = \frac{kT}{(\gamma-1)m_p}$$

$$\partial_t \rho + \vec{\nabla}.(\rho\vec{V}) = 0$$

$$\partial_t(\rho\vec{V}) + \vec{\nabla}(\rho\vec{V}\vec{V} - \vec{B}\vec{B} + P_{tot}I) = 0$$

$$\partial_t E + \vec{\nabla}.((E+P)\vec{V} - \vec{B}(\vec{B}.\vec{V})) = 0$$

$$\partial_t \vec{B} + \vec{\nabla}\times(\vec{B}\times\vec{V}) = 0$$

$$\vec{\nabla}.\vec{B} = 0$$

where

$$E = \rho e + \frac{1}{2}\rho\vec{V}^2 + \frac{1}{2}\vec{B}^2$$

$$P_{tot} = P + \frac{1}{2}\vec{B}^2$$

In 1D can also be written as:

$$\partial_t U + \partial_x F = \partial_t U + A \partial_x U = 0$$

$$U = (\rho, \rho u, \rho v, \rho w, B_y, B_z, E)$$

$$F = \begin{cases} \rho u \\ \rho u^2 + P_T - B_x{}^2 \\ \rho u v - B_x B_y \\ \rho u w - B_x B_z \\ B_y u - B_x v \\ B_z u - B_x w \\ (E + P_T) u - B_x (u B_x + v B_y + w B_z) \end{cases}$$

$$B_x = cst$$

A is called the Jacobian

Its eigenvalues are the wave speeds.

In 3D, we have:

$$\partial_t U + \partial_x F + \partial_y G + \partial_z H = 0$$

$$U = (\rho, \rho u, \rho v, \rho w, B_x, B_y, B_z, E)$$

$$\vec{\nabla} . \vec{B} = 0$$

Brief description of the MHD waves

The 1D MHD equations have seven eigenvalues or equivalently give rise to 7 waves.

2 Alfvén waves: transverse mode (analogous to the vibration of a string)

2 slow magneto-acoustic waves (coupling between Lorentz force and thermal pressure, B and ρ are anticorrelated)

2 fast magneto-acoustic waves (coupling between Lorentz force and thermal pressure, B and ρ are correlated)

1 entropy wave (contact discontinuity, does not propagate)

$$\lambda_{2,6} = u \pm c_a, \lambda_{1,7} = u \pm c_f, \lambda_{3,5} = u \pm c_s, \lambda_4 = u,$$

$$where$$

$$c_a = \frac{B_x}{\sqrt{\rho}}, c_{f,s} = \left(\frac{\gamma p + \vec{B}^2 \pm \sqrt{(\gamma p + \vec{B}^2)^2 - 4\gamma p B_x^{\ 2}}}{2\rho} \right)^{1/2}$$

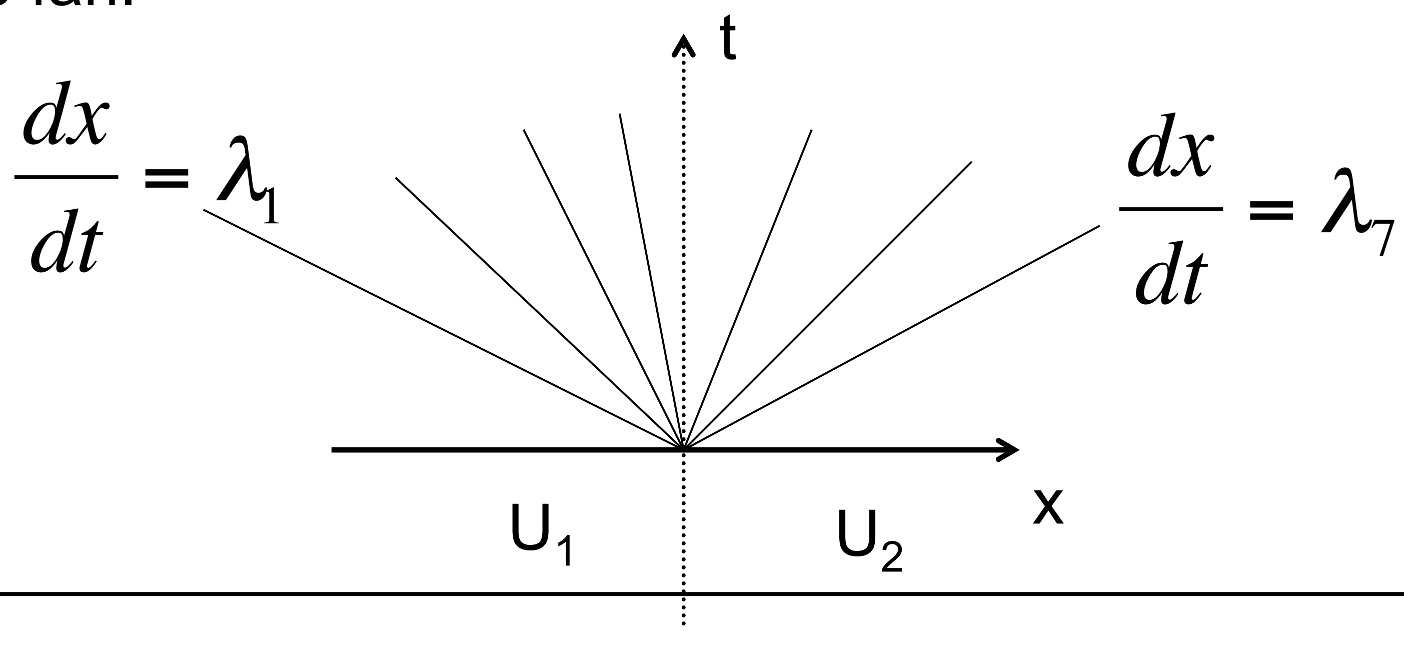
The wave velocities are such that:

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7$$

fast - Alfvén - slow - entropy - slow - Alfvén - fast

Therefore, some eigenvalues may coincide depending on the direction and the strength of the magnetic field (whereas hydro case is strickly hyperbolic).

Wave fan:

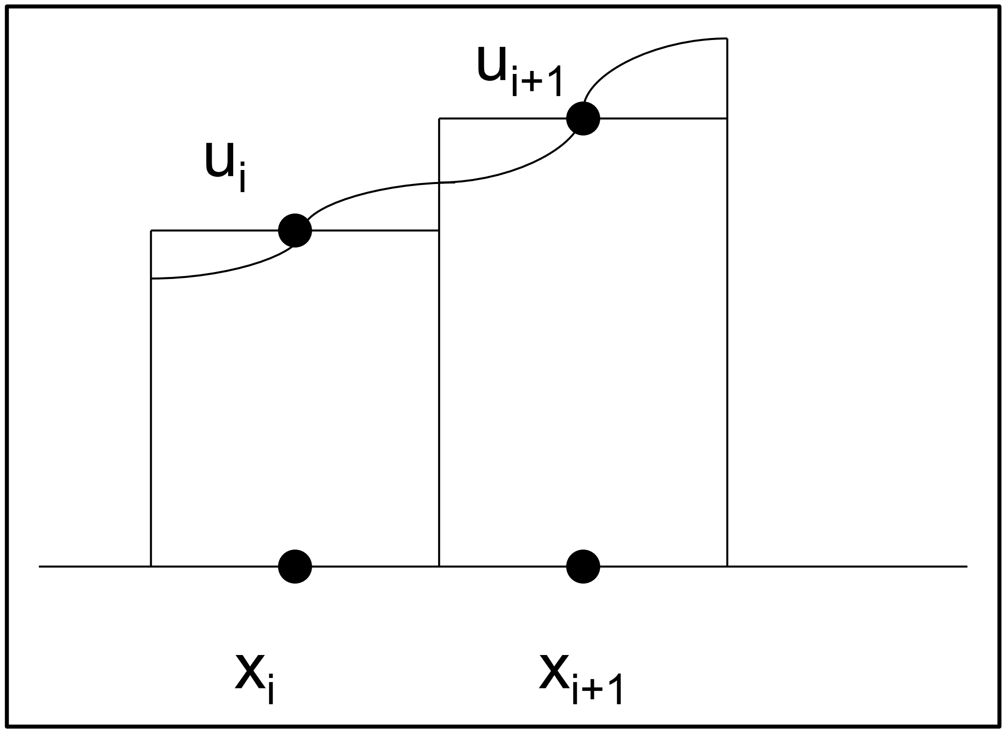


Godunov type methods

Originally developed to solve compressible hydrodynamical equations (Godunov 1959). Well suited to handdle shocks and discontinuities

=> This is why they are so commonly used in astrophysics. No need to introduce viscosity to stabilize the scheme. Discontinuities resolved in few cells.

Each computational cell represents a fluid volume with uniform density, velocity, energy inside the cell which represents the average values.



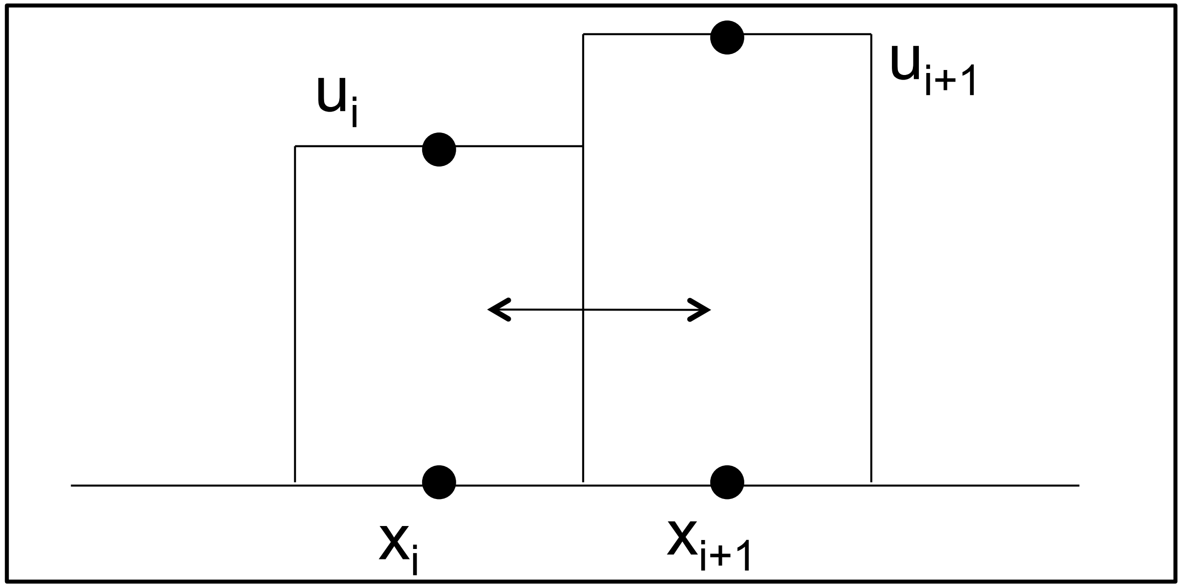
The total mass, momentum and energy within the cells are thus

$$dM_{cell} = \overline{\rho}_i dx(dydz), d\vec{P}_{cell} = \vec{P}_i dx(dydz), dE_{cell} = \overline{E}_i dx(dydz),$$

$$with\ \overline{U}_i(t) = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(t,x')dx' \quad (in\ 1D)$$

$$\overline{U}_{i,j,k}(t) = \frac{1}{\Delta x \Delta y \Delta z}\int_{x_{i-1/2},y_{i-1/2},z_{i-1/2}}^{x_{i+1/2},y_{i+1/2},z_{i+1/2}} dx'dy'dz'U(t,x',y',z') \quad (in\ 3D)$$

The cells exchange flux of matter, momentum and energy between each others. *Philosophy* different from finite difference methods, in which the discrete values represent the exact values of the quantity at the location, or from Fourier methods.



The *exact* discretized solution of $\partial_t U + \partial_x F = 0$ is given by:

$$\overline{U}_i(\Delta t) = \overline{U}_i(0) + \frac{\Delta t}{\Delta x}\left(\overline{F}(x_i) - \overline{F}(x_{i+1})\right)$$

$$where\ \overline{F}(x_i) = \frac{1}{\Delta t}\int_0^{\Delta t} dt\ F(U(t,x_i))\ (in\ 1D)$$

$$\overline{F}_{j,k}(x_i) = \frac{1}{\Delta t \Delta y \Delta z}\int_0^{\Delta t} dt \int_{y_j}^{y_{j+1}} dy \int_{z_k}^{z_{k+1}} dz\ F(U(t,x_i,y,z))\ (in\ 3D)$$

Note that:

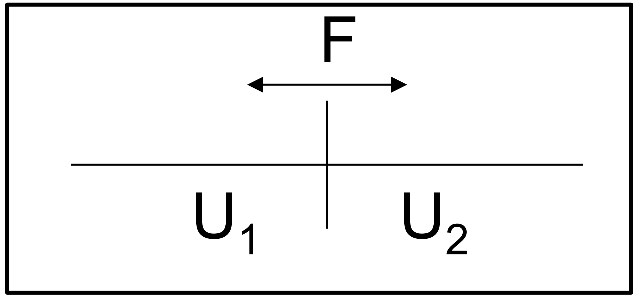
-this is at this stage an exact solution, in practice however the fluxes are approximately calculated

-even if the fluxes are not correct, the method, *by construction*, conserves mass, momentum and energy exactly since the amount retrieve from one cell is exactly given to its neighbour

-this expression does not entail derivative but flux differences, this is why, discontinuities are well resolved. This is unlike finite difference methods or spectral methods.

The Riemann Problem

The question with the Godunov method is thus to estimate accurately the fluxes exchanged between two uniform states U_1 and U_2.



This is called the Riemann problem.

Since no characteristic scale is involved in the problem, it is *self-similar*. That is to say the pattern at (x,t) can be deduced from the pattern at (x' ,t'), U(x/t)=U(x' /t'). Thus, the flux exchanged between the 2 states is constant in time.

A central problem for Godunov type methods, is to have accurate « Riemann solvers » which resolve the Riemann problem at interface between cells and provide the flux.

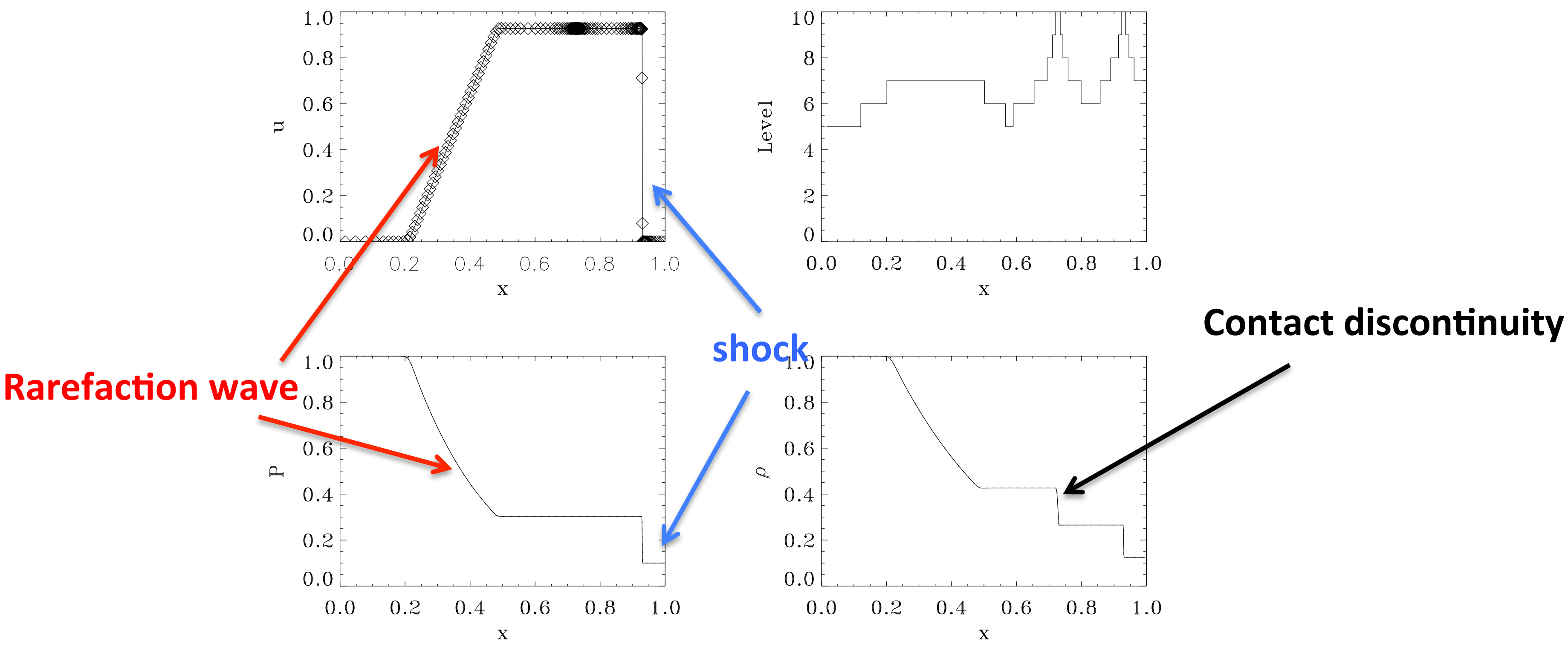
Solving the Riemann problem for non linear equations is in general a very difficult problem.

RIEMANN SOLVER

1D MHD

Exact hydrodynamical Riemann solver

-Hydrodynamical Riemann problem entails 3 non-linear waves, rarefaction wave, contact discontinuity and shock



-Exact hydrodynamical solver is known

-Need to perform several iterations

=>Accurate but expensive

=>Interest in having cheaper solvers

In MHD, no exact solver is known (would be very expensive)

=> Need to find approximate solvers

Solution of the Riemann Problem for linear hyperbolic systems

(Toro 1999)

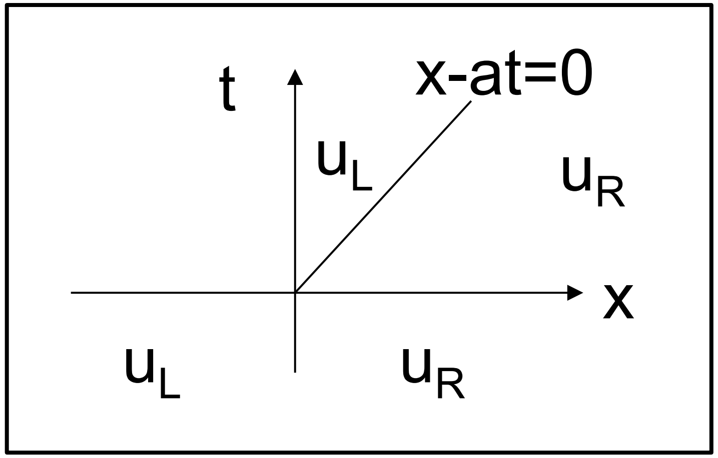
Finding the solution of the Riemann problem is possible when the Jacobian A of the system is a constant matrix. As will be seen later, this turns out to be extremely useful.

Let us consider the simple linear advection equation: $\partial_t u + a\partial_x u = 0$

The solutions are simply given by: $f(x - at)$

Thus the rate of change of u along the characteristic curve dx/dt=a is zero. a is called the characteristic speed.

In this case the solution of the Riemann problem is very simple:



$$x - at < 0 \Rightarrow u = u_L$$

$$x - at > 0 \Rightarrow u = u_R$$

Now let us consider a linear system of m variables and equations.

$$\partial_t U + A\partial_x U = 0, A, \text{ a constant } m\times m \text{ matrix.}$$

Let us diagonalise A:

$$A = K^{-1}\Lambda K, \Lambda = \begin{bmatrix} \lambda_1 & . & . & . & 0 \\ 0 & . & & & . \\ . & & . & & . \\ . & & & \lambda_{n-1} & . \\ 0 & . & . & . & \lambda_n \end{bmatrix}, \quad K = [K^1,...,K^m], \quad AK^i = \lambda_i K^i$$

$$W = K^{-1}U \Rightarrow U = KW, \quad K^{-1}\partial_t U + K^{-1}A\partial_x U = \partial_t W + \Lambda\partial_x W = 0$$

Thus, the system is decoupled and the solution fo each W_i is just W_i $(x-\lambda_i t)$.
Coming back to U, we have: $U(x,t) = \sum_{i=1,m} W_i(x-\lambda_i t)K^i$

Let us consider again the Riemann problem, $U(x,0)=U_L$ for $x<0$ and U_R for $x>0$
We can write:

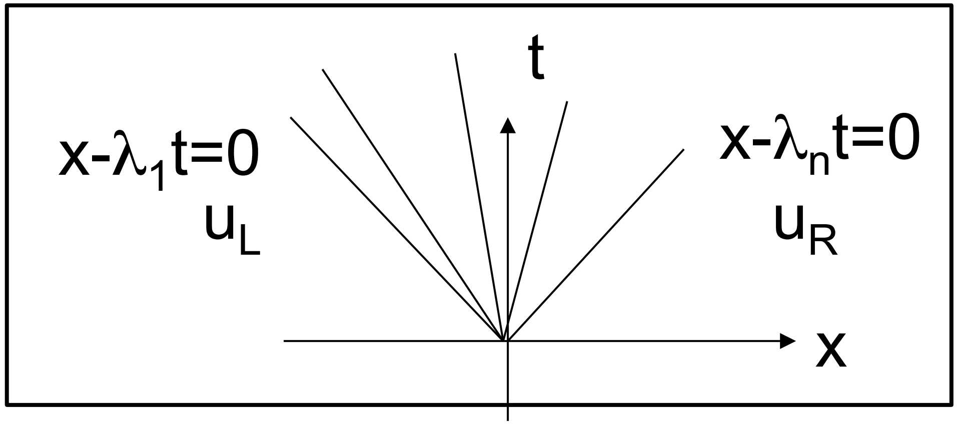
$$U_L = \sum_{i=1,m}\alpha_i K^i, \quad U_R = \sum_{i=1,m}\beta_i K^i$$

$$W_i = \alpha_i \text{ if } x<0, \quad W_i = \beta_i \text{ if } x>0.$$

For a given (x,t), there is an eigenvalue such that $\lambda_I < x/t < \lambda_{I+1}$:

$$U(x,t) = \sum_{i=I+1,m}\alpha_i K^i + \sum_{i=1,I}\beta_i K^i$$

Thus, we have the following picture:



The solution consists of m waves emanating from the origin.They constitute the wave fan. Each wave carries a jump discontinuity.

We can now estimate the flux exchanged between the cells that will be needed to advance the solution. The value of U at the interface is U(0):

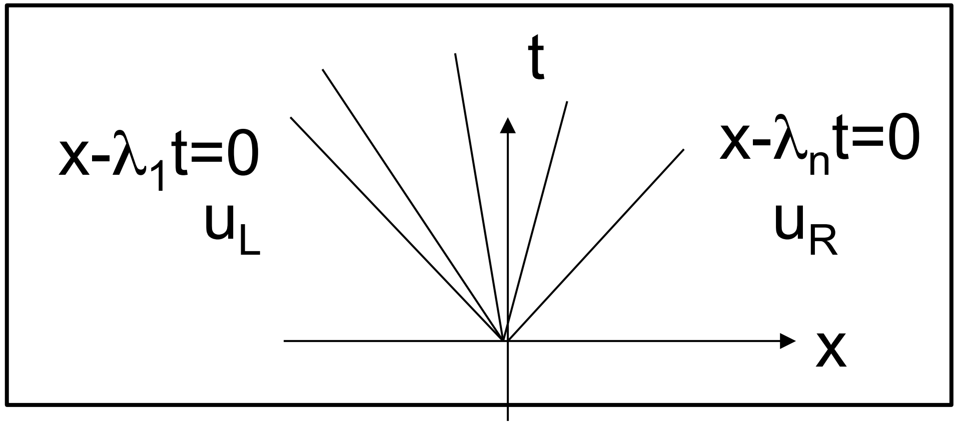
$$U(0) = \sum_{i=I+1,m} \alpha_i K^i + \sum_{i=1,I} \beta_i K^i = \sum_{i=1,m} \alpha_i K^i + \sum_{i=1,I} (\beta_i - \alpha_i) K^i$$

$$= U_L + \sum_{i=1,I} (\beta_i - \alpha_i) K^i = U_R - \sum_{i=I+1,m} (\beta_i - \alpha_i) K^i = \frac{1}{2}(U_L + U_R) + \frac{1}{2}\sum_{i=1,m} sign(\lambda_i)(\beta_i - \alpha_i) K^i$$

Thus, we can obtain the Godunov flux:

$$F(U(0)) = A(U(0)) = \frac{1}{2}(A(U_L) + A(U_R)) + \frac{1}{2}\sum_{i=1,m} sign(\lambda_i)(\beta_i - \alpha_i) A K^i$$

$$= \frac{1}{2}(F_L + F_R) + \frac{1}{2}\sum_{i=1,m} |\lambda_i| (\beta_i - \alpha_i) K^i$$



$$F = \underbrace{\frac{1}{2}(F_L + F_R)}_{\text{Mean flux}} + \underbrace{\frac{1}{2}\sum_{i=1,m}\left|\lambda_i\right|(\beta_i - \alpha_i)K^i}_{\text{Diffusive part ensuring upwinding and code stability}}$$

An example:

Let us consider the linearized 1D hydrodynamical equations.

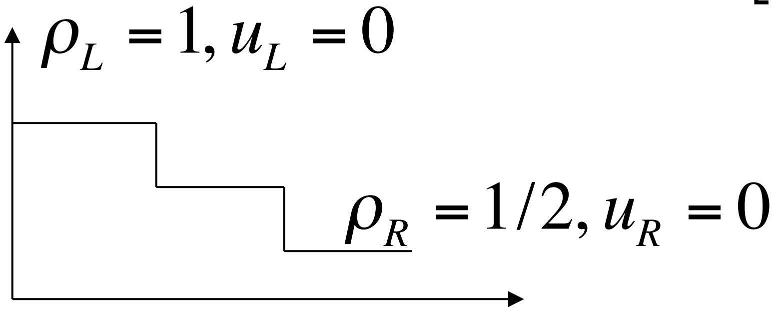
$$U_t + AU_x = 0,$$

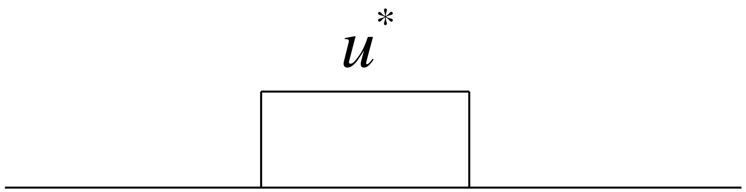
$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \rho \\ u \end{bmatrix}, A = \begin{bmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{bmatrix} \rightarrow \lambda_1 = -a, \lambda_2 = a, K^{(1)} = \begin{bmatrix} \rho_0 \\ -a \end{bmatrix}, K^{(2)} = \begin{bmatrix} \rho_0 \\ a \end{bmatrix}$$

$$U_L = \begin{bmatrix} \rho_L \\ u_L \end{bmatrix} = \alpha_1 K^{(1)} + \alpha_2 K^{(2)} \Rightarrow \alpha_1 = \frac{a\rho_L - \rho_0 u_L}{2a\rho_0}, \alpha_2 = \frac{a\rho_L + \rho_0 u_L}{2a\rho_0},$$

$$idem(\beta, R) \leftrightarrow (\alpha, L)$$

$$U^* = \begin{bmatrix} \rho^* \\ u^* \end{bmatrix} = \alpha_2 K^{(2)} + \beta_1 K^{(1)} = \begin{bmatrix} \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2}(u_R - u_L)\rho_0/a \\ \frac{1}{2}(u_L + u_R) - \frac{1}{2}(\rho_R - \rho_L)a/\rho_0 \end{bmatrix}$$





The ROE Riemann solver (MHD solver)

3 waves linear solver for HD (Roe 1981, Toro 1999).

7 waves linear solver for MHD (Brio & Wu 1988, Cargo & Gallice 1998, Balsara 1998).

Complex method which requires some calculations. Only the basic ideas presented here.

Solving the Riemann problem exactly is too difficult so one replaces the non linear problem by a linear problem that is solved *exactly*.

Replace the new Jacobian, A, by a linear one which has adequate properties.

$$\partial_t U + F(U)_x = 0, \partial_t U + AU_x = 0 \approx \partial_t U + \tilde{A}U_x = 0, \quad \tilde{A}(U_L, U_R)$$

It is required to have the following properties:

Property (A): Hyperbolicity of A, implying that it has m eigenvalues and eigenvectors. $\tilde{\lambda}_1 \leq \ldots \leq \tilde{\lambda}_m, \tilde{K}_1, \ldots, \tilde{K}_m,$

This preserves the linear wave structure of the original problem.

Property (B): $\tilde{A}(U,U) = A(U)$

This is called the *consistency*. It implies that in the limit where the right and left states becomes identical, the flux is exactly recovered.

Property (C): $\tilde{A}(U_R,U_L)(U_R - U_L) = F(U_R) - F(U_L)$

This is the most difficult property to satisfy.

It ensures that an isolated discontinuity which satisfies the jump relation:

$$F_R - F_L = \lambda_c (U_R - U_L)$$

will be adequatly described by the solver (projected in a single eigenvector giving $\lambda_c=\lambda_i$).

Constructing a Roe matrix is not easy. Simple averaging like $0.5(A(U_R)+A(U_L))$ does not verify property (C).

This can be achieved (Roe 1981) by introducing an intermediate vector Q.

$$U = U(Q), F = F(Q), Q = \sqrt{\rho}(1,u,v,w,H,B_y/\rho,B_z/\rho)$$

$$H = \frac{1}{\rho}\left(E + P + \frac{1}{2}\left(B_x^{\,2} + B_y^{\,2} + B_z^{\,2}\right)\right)$$

By doing this, it is found that U and F express as algebraic relations (product Q_iQ_j or ratio Q_i/Q_j) involving the components of Q. But we have for example:

$$Q_{1,R}Q_{2,R} - Q_{1,L}Q_{2,L} = \Delta(Q_1Q_2) = \overline{Q}_2\Delta Q_1 - \overline{Q}_1\Delta Q_2$$

Thus the jump relations can be expressed by the jump relation of Q:

$$F_R - F_L = \Delta F = \tilde{C}\Delta Q, \quad (U_R - U_L) = \Delta U = \tilde{B}\Delta Q$$

Thus, $$\tilde{A} = \tilde{C}\tilde{B}^{-1}, \quad F_R - F_L = \tilde{A}(U_R - U_L)$$

And the flux is given by the formula obtained previously:

$$F(U(0)) = \frac{1}{2}(F_L + F_R) + \frac{1}{2}\sum_{i=1,m}\left|\tilde{\lambda}_i\right|(\tilde{\beta}_i - \tilde{\alpha}_i)\tilde{K}^i$$

To summarize, the whole algorithm is:

-compute the Roe average, involved quantities like: $$\tilde{u} = \frac{\sqrt{\tilde{\rho}_L}u_L + \sqrt{\tilde{\rho}_R}u_R}{\sqrt{\tilde{\rho}_L} + \sqrt{\tilde{\rho}_R}}$$

-compute the eigenvalues and eigenvectors

-compute the wave strength ($\alpha-\beta$)

-compute the flux

Generally speaking, the Roe solver works well and gives accurate results. It is widely used and serves as a reference.

In some rare occasions (but not so rare….), the Roe solver is encountering severe difficulties and crashes.This is due to the linearisation which is a poor approximation for highly non linear discontinuities encountered in stiff problems.

The manifestation of this can be:

-intermediate states with negative energy or density

-rarefaction shocks leading to entropy violation

An entropy fix or more generally a switch is needed to cure these events… Various possibilities have been proposed (see e.g. Toro 1999).

For example, one can switch to HLL using the largest and smallest wave speed of Roe. This replaces the 6 intermediate Roe states by a single star state.

Shock Jump conditions

Across a discontinuity (that is to say in any point), and in the frame moving with it, jump conditions apply: $F_1 = F_2$

In the laboratory frame, the discontinuity is moving at some speed λ_C. The jump relation can then be written as:

$$\lambda_c U_1 - F_1 = \lambda_c U_2 - F_2$$

To see this, let us consider again the equation: $\partial_t U + \partial_x F = 0$ and a control volume [X_L,X_R]. A corresponding integral form on the volume of control is:

$$F(U_R) - F(U_L) = \frac{d}{dt}\int_{X_L}^{x(t)} U(x,t)dt + \frac{d}{dt}\int_{x(t)}^{X_R} U(x,t)dt$$

$$= \frac{dx}{dt}U(x(t)_-,t) - \frac{dx}{dt}U(x(t)_+,t) + \int_{X_L}^{x(t)} \partial_t U(x,t)dt + \int_{x(t)}^{X_R} \partial_t U(x,t)dt$$

Thus, if X_L->X_R, the integrals on the right hand side vanish and we obtain the relation.

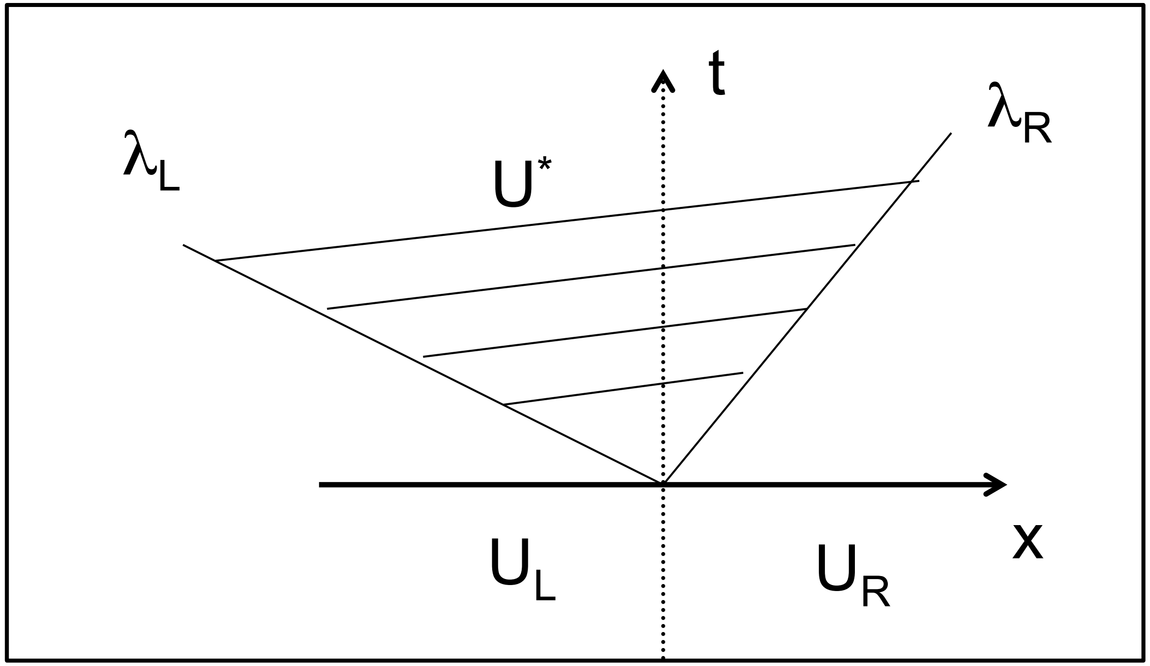
The H(arten)L(ax) (van)L(eer) Riemann solver

(Harten et al. 1983, Toro 1999)

2 waves solver (hydro and mhd):

one retains only the 2 fastest waves (e.g. the 2 fast magneto-accoustic waves) and then assume that between the 2 waves, there is a uniform state U*.

Conservation laws are then used to determine U* and the flux F *.



First step (HLL)
Let us consider a volume of control V, i.e. an area of surface S in YZ and delimited by -L and L in X.

At time t=0, the total value of U within V is: $(S \times 2L) \times U_{tot} = S \times L \times (U_L + U_R)$

At time t, the left and the right waves have reached: $X = \lambda_L \times t, X = \lambda_R \times t$

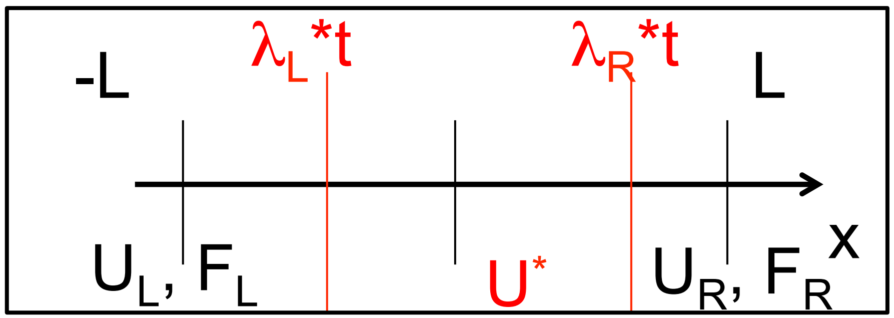
Thus:

$$(S \times 2L) \times U_{tot}(t) =$$
$$S \times ((L + \lambda_L t) \times U_L + (L - \lambda_R t) \times U_R + (-\lambda_L + \lambda_R) t \times U^*)$$

But we also have: $(S \times 2L) \times (U_{tot}(t) - U_{tot}(0)) = (F_L - F_R) \times t$

Thus we obtain U*:

$$U^* = \frac{F_L - F_R + \lambda_R U_R - \lambda_L U_L}{\lambda_R - \lambda_L}$$



Second step (HLL)

But what we want, is to determine F*, so the job is not finished yet

(F(U*) is not a good solution). Assume first that: $\lambda_L<0$, $\lambda_R>0$

Let us consider a new volume of control, delimited by X=-L and X=0. Then we have:

$$S\times((L+\lambda_L t)\times U_L-\lambda_L t\times U^*)=SL\times U_L+(F_L-F^*)t$$

and thus:

$$F^*=F_L+\lambda_L(U^*-U_L)=\frac{\lambda_R F_L-\lambda_L F_R+\lambda_L\lambda_R(U_R-U_L)}{\lambda_R-\lambda_L}$$

Note that this expression is symmetrical in R <=> L indicating that we could have used X=0, X=L as volume of control and find the same result.

If now we assume that: $\lambda_L>0$, $\lambda_R>0$, that is to say the left state propagates faster than the fastest wave in the right direction, the same calculation shows that $F_{HLL}=F_L$. In the same way $\lambda_L<0$, $\lambda_R<0$ implies $F_{HLL}=F_R$

$$\lambda_L<0,\lambda_R>0\rightarrow F_{HLL}=F^*=\frac{\lambda_R F_L-\lambda_L F_R+\lambda_L\lambda_R(U_R-U_L)}{\lambda_R-\lambda_L}$$

$$\lambda_L>0,\lambda_R>0\rightarrow F_{HLL}=F_L$$

$$\lambda_L<0,\lambda_R<0\rightarrow F_{HLL}=F_R$$

$$\textit{Note that}:\quad F_L\rightarrow F^*\ \textit{when}\ \lambda_L\rightarrow 0$$

Which wave speed ?

In principle, determining the correct wave speeds would require to solve the problem exactly first… Fortunately, good estimates can be made.

Davis (1988) propose:

$$S_L = \min[\lambda_l(U_L),\lambda_l(U_R)]$$
$$S_R = \max[\lambda_m(U_L),\lambda_m(U_R)]$$

while Einfeldt et al. (1991) propose:

$$S_L = \min[\lambda_l(U_L),\lambda_l(U_{Roe})]$$
$$S_R = \max[\lambda_m(U_L),\lambda_m(U_{Roe})]$$

where λ_l and λ_m are respectively the smallest and largest wave speeds and λ_{roe} are the Roe wave speeds.

Positivity of the scheme

In the hydrodynamical case, the scheme ensures positivity that is to say, density and pressure remain positive (Einfeldt et al. 1991). The common experience is that the scheme is very robust.

However, the scheme does not resolve contact discontinuities and is therefore very diffusive. Single-state approximation should be extended to a two or multi-state approximation.

Note when $\lambda_L=\lambda_R$ is enforced, the scheme is called Lax-Friedrich solver.