The RAMSES code and related techniques

2- MHD solvers

Romain Teyssier

University of Zurich

irfu

cea

saclay

Outline

- The ideal MHD equations
- Godunov method for 1D MHD equations
- Ideal MHD in multiple dimensions
- Cell-centered variables: divergence B cleaning
- Face-centered variables: the CT approach
- A few MHD numerical tests

The ideal MHD equations

Mass conservation $$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation $$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + P) - \mathbf{j} \times \mathbf{B} = 0$$

Energy equation $$\partial_t (\rho \epsilon) + \nabla \cdot (\rho \epsilon \mathbf{u}) + P \nabla \cdot \mathbf{u} = 0$$

Induction equation $$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

Ampère's law $$\nabla \times \mathbf{B} = 4\pi \mathbf{j}$$

No magnetic monopoles $$\nabla \cdot \mathbf{B} = 0$$

The ideal MHD equations in conservative forms

Mass conservation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} - \frac{1}{4\pi}\mathbf{B}\mathbf{B}) + \nabla P_{tot} = 0$$

Total energy conservation

$$\partial_t E + \nabla \cdot \left[(E + P_{tot})\mathbf{u} - \frac{1}{4\pi}\mathbf{B}(\mathbf{B} \cdot \mathbf{u}) \right] = 0$$

Magnetic flux conservation

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0$$

Total energy

$$E = \rho\epsilon + \frac{1}{2}\rho \mathbf{u}^2 + \frac{1}{8\pi}\mathbf{B}^2$$

Total pressure

$$P_{tot} = P + \frac{1}{8\pi}\mathbf{B}^2$$

No magnetic monopoles

$$\nabla \cdot \mathbf{B} = 0$$

The ideal MHD equations in 1D

For 1D (plane symetric flow), one has $\nabla \cdot \mathbf{B} = 0 \rightarrow B_x = \text{constant}$

The vector of conservative variables writes $\mathbf{U} = (\rho, \rho u, \rho v, \rho w, B_y, B_z, E)$

Ideal MHD in conservative form: $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$

Flux function

$$\mathbf{F} = \left| \begin{array}{l} \rho u \\ \rho u^2 + P_{tot} - B_x^2 \\ \rho u v - B_x B_y \\ \rho u w - B_x B_z \\ B_y u - B_x v \\ B_z u - B_x w \\ (E + P_{tot}) u - B_x (B_x u + B_y v + B_z w) \end{array} \right.$$

MHD waves

Compute the Jacobian matrix $\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$

It has 7 real eigenvalues (ideal MHD equations are hyperbolic), one for each wave:

2 fast magnetosonic waves: $\lambda_1 = u - c_f$ $\lambda_7 = u + c_f$

2 Alfven waves: $\lambda_2 = u - c_a$ $\lambda_6 = u + c_a$

2 slow magnetosonic waves: $\lambda_3 = u - c_s$ $\lambda_5 = u + c_s$

1 entropy waves: $\lambda_4 = u$

Fast magnetosonic waves are longitudinal waves with variations in pressure and density (correlated with magnetic field)

Slow magnetosonic waves are longitudinal waves with variations in pressure and density (anti-correlated with magnetic field)

Alfven waves are transverse waves with no variation in pressure and density.
Entropy wave is a contact discontinuity with no variation in pressure and velocity.

MHD wave speed

Sound speed: $c_0^2 = \frac{\gamma P}{\rho}$ Alfven speed: $c_{a,x}^2 = \frac{B_x^2}{4\pi\rho}$ $c_a^2 = \frac{B^2}{4\pi\rho}$

Fast magnetosonic speed: $c_f^2 = \frac{1}{2}(c_0^2 + c_a^2) + \frac{1}{2}\sqrt{(c_0^2 + c_a^2)^2 - 4c_0^2 c_{a,x}^2}$

Slow magnetosonic speed: $c_s^2 = \frac{1}{2}(c_0^2 + c_a^2) - \frac{1}{2}\sqrt{(c_0^2 + c_a^2)^2 - 4c_0^2 c_{a,x}^2}$

$$u - c_f < u - c_a < u - c_s < u < u + c_s < u + c_a < u + c_f$$

In some special cases, some wave speeds can be equal:

The ideal MHD system is not strictly hyperbolic.

This can lead to exotic features such as *compound waves*

(for example a mixture of Alfven wave and shock)

Godunov method for 1D MHD flows

Godunov methodology applies to any hyperbolic system of conservation laws

Stability is ensured by proper upwinding of the flux function with respect to the 7 MHD waves.

Second order accuracy is obtained by a predictor-corrector approach (the MUSCL scheme).

MHD equations are solved in conservative form: this ensures that Rankine-Hugoniot relations are satisfied (proper shock jump conditions).

use the Jacobian   use slope limiters

Step 1: predictor step

$$\mathbf{W}^{n+1/2}_{i+1/2,L} = \mathbf{W}^n_i + \frac{\Delta t}{2}\left(\frac{\partial \mathbf{W}}{\partial t}\right)_i + \frac{\Delta x}{2}\left(\frac{\partial \mathbf{W}}{\partial x}\right)_i$$

Step 2: compute flux

$$\mathbf{F}^{n+1/2}_{i+1/2} = \mathbf{F}^*(\mathbf{W}^{n+1/2}_{i+1/2,L}, \mathbf{W}^{n+1/2}_{i+1/2,R})$$

← Riemann solver

Step 3: conservative update

$$\frac{\mathbf{U}^{n+1}_i - \mathbf{U}^n_i}{\Delta t} + \frac{\mathbf{F}^{n+1/2}_{i+1/2} - \mathbf{F}^{n+1/2}_{i-1/2}}{\Delta x} = 0$$

The Roe Riemann solver for MHD

It belongs to the more general class of *linear Riemann solvers*.

Generalise the Euler 3 wave Roe solver to a seven wave MHD Roe solver (Brio & Wu 1988; Cargo & Gallice 1989)

Define the Roe average state: $\mathbf{U}_{ref} = \mathrm{Roe}\left[\mathbf{U}_L, \mathbf{U}_R\right]$

$$\bar{\rho} = \sqrt{\rho_L}\sqrt{\rho_R} \qquad \bar{\mathbf{u}} = \frac{\sqrt{\rho_L}\mathbf{u}_L + \sqrt{\rho_R}\mathbf{u}_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \qquad \bar{B}_\perp = \frac{\sqrt{\rho_R}B_{\perp L} + \sqrt{\rho_L}B_{\perp R}}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

Compute the Jacobian matrix for this reference state $\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}\left(\mathbf{U}_{ref}\right)$

The Roe average is computed in order to get *Property (C)*:

$$\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) = \mathbf{A}(\mathbf{U}_{ref})\left(\mathbf{U}_R - \mathbf{U}_L\right)$$

Eigenvector decomposition of the Roe matrix: $\mathbf{A} = \mathbf{L}^T \Lambda \mathbf{R} \qquad |\Lambda| = (|\lambda_1|, |\lambda_2|, \ldots)$

The Roe flux is defined as: $\mathbf{F}^*(U_L, U_R) = \frac{\mathbf{F}_L + \mathbf{F}_R}{2} - \mathbf{L}^T|\Lambda|\mathbf{R}\frac{\mathbf{U}_R - \mathbf{U}_L}{2}$

The Roe solver is the only 7 waves Riemann solver to date. It is a bit slow and features one problem: formation of rarefaction shocks.

The HLL Riemann solvers for MHD

HLL-type Riemann solvers rely only on computing the fastest wave speed.

Define *a* as the fast magnetosonic speed and the left and right going waves as

$$S_L = \min(u_L, u_R) - \max(a_L, a_R) \qquad S_R = \max(u_L, u_R) + \max(a_L, a_R)$$

Use generic Rankine-Hugoniot relations with one single intermediate state *U** and corresponding flux *F*,* we get the HLL flux:

$$S_L > 0 \quad \mathbf{F}^*(U_L, U_R) = \mathbf{F}_L$$

$$S_R < 0 \quad \mathbf{F}^*(U_L, U_R) = \mathbf{F}_R$$

$$S_L < 0 \ \text{ and } \ S_R > 0 \quad \mathbf{F}^*(U_L, U_R) = \frac{S_R \mathbf{F}_L - S_L \mathbf{F}_R + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L}$$

The Lax-Friedrich flux is obtained as a particular case with $S_* = S_R = -S_L$

$$\mathbf{F}^*(U_L, U_R) = \frac{\mathbf{F}_L + \mathbf{F}_R}{2} - S_* \frac{\mathbf{U}_R - \mathbf{U}_L}{2}$$

The HLLC Riemann solver

Generalise Euler HLLC to MHD (Linde 2002; Gurski 2004)

Define the Lagrangian sound speed as: $c_R = \rho_R(S_R - u_R)\ \ c_L = \rho_L(u_L - S_L)$

Define P_{tot}* and u* as:

Thermal + magnetic pressure

$$P^* = \frac{c_R P_L + c_L P_R + c_L c_R (u_L - u_R)}{c_L + c_R}$$

$$u^* = \frac{c_L u_L + c_R u_R + P_L - P_R}{c_L + c_R}$$

← This defines the 3rd contact wave

Left and right density jumps: $\rho_L^* = \rho_L \frac{S_L - u_L}{S_L - u^*}$ $\rho_R^* = \rho_R \frac{S_R - u_R}{S_R - u^*}$

For transverse velocity and magnetic field, we get:

$$v_L^* = v_L \quad w_L^* = w_L \quad B_{y,L}^* = B_{y,L} \frac{S_L - u_L}{S_L - u^*} \quad B_{z,L}^* = B_{z,L} \frac{S_L - u_L}{S_L - u^*}$$

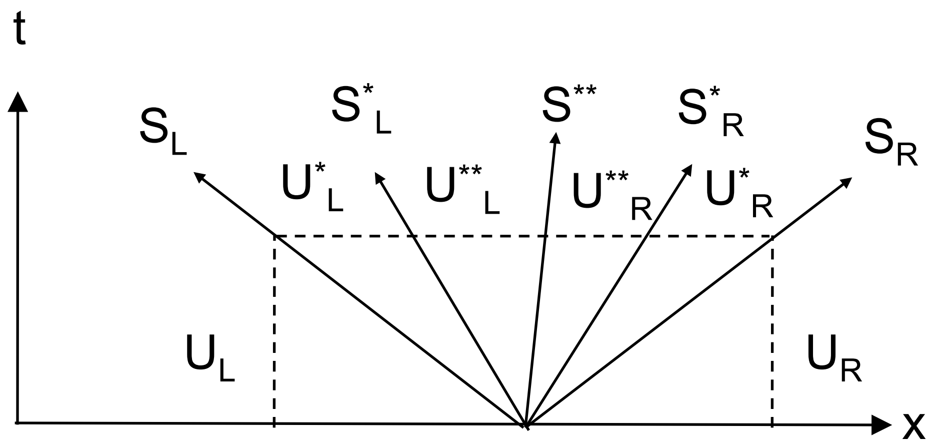
Finally, compute fluxes using Rankine-Hugoniot relations on the 3 discontinuities.

Jump conditions are not correct (especially at the contact discontinuity).

Valid only for B_x=0. Otherwise, HLLC can lead to numerical instability or excessive smoothing of Alfven waves.

The HLLD Riemann solver

5 wave Riemann solver with 4 intermediate (star) states (Miyoshi & Kusano 2005)



Normal velocity and pressure are uniform across 4 states (HLLC values).

Density jumps across fast waves but is uniform across Alfven waves (HLLC values)

Define the two new wave speeds as the star state Alfven speed:

$$S_L^* = u^* - \frac{|B_x|}{\sqrt{\rho_L^*}} \qquad S_R^* = u^* + \frac{|B_x|}{\sqrt{\rho_R^*}}$$

Given the 5 wave speed, we can compute the intermediate states using Rankine-Hugoniot relations.

The HLLD Riemann solver

RH relations after the fast magnetosonic wave give jumps in transverse v and B

$$B^*_{\perp,L} = B_{\perp,L}\frac{\rho_L(S_L-u_L)^2-B_x^2}{\rho_L(S_L-u_L)(S_L-u^*)-B_x^2}$$

$$v^*_{\perp,L} = v_{\perp,L} - B_x B_{\perp,L}\frac{u^*-U_L}{\rho_L(S_L-u_L)(S_L-u^*)-B_x^2}$$

Conservation laws across the 2 Alfven waves give the following uniform values (only density varies across the contact waves)

$$B^{**}_{\perp} = \frac{\sqrt{\rho^*_L}B^*_{\perp R} + \sqrt{\rho^*_R}B^*_{\perp L} + \sqrt{\rho^*_L}\sqrt{\rho^*_R}(v^*_{\perp R}-v^*_{\perp L})\mathrm{sign}(B_x)}{\sqrt{\rho^*_L}+\sqrt{\rho^*_R}}$$

$$v^{**}_{\perp} = \frac{\sqrt{\rho^*_L}v^*_{\perp L} + \sqrt{\rho^*_R}v^*_{\perp R} + (B^*_{\perp R}-B^*_{\perp L})\mathrm{sign}(B_x)}{\sqrt{\rho^*_L}+\sqrt{\rho^*_R}}$$

Note that these relations are HLL states applied to the 2 Alfven waves.

The HLLD Riemann solver

Flux for each conservative variable are computed using RH relations across each waves.

As for HLLC in the case of the Euler equations, it can be shown that HLLD is a positivity preserving scheme.

It resolves exactly fast magnetosonic and Alfven (rotational) waves, as well as contact discontinuities.

Slow magnetosonic waves are excessively smoothed.

Fast and robust Riemann solver, now widely used (RAMSES, ATHENA, ENZO).

A good strategy:

1- try first HLLD

2- if it doesn't work, try then HLL

3- if it doesn't work, try Lax-Friedrich (it should work !!!)

MHD shock tube test

Left and right states defined by $\mathbf{W} = (\rho, P, u, v, w, B_x/\sqrt{4\pi}, B_y/\sqrt{4\pi}, B_z/\sqrt{4\pi})$

$$\mathbf{W}_L = (1.08, 0.95, 1.2, 0.01, 0.5, 4/\sqrt{4\pi}, 3.6/\sqrt{4\pi}, 2/\sqrt{4\pi})$$
$$\mathbf{W}_R = (1.0, 1.0, 0.0, 0.0, 0.0, 4/\sqrt{4\pi}, 4/\sqrt{4\pi}, 2/\sqrt{4\pi})$$

```
&INIT_PARAMS
nregion=2
region_type(1)='square'
region_type(2)='square'
x_center=0.25,0.75
length_x=0.5,0.5
d_region=1.08,1.0
u_region=1.2,0.0
v_region=0.01,0.0
w_region=0.5,0.0
p_region=0.95,1.0
A_region=1.12838,1.12838
B_region=1.01554,1.12838
C_region=0.564190,0.564190
/

&OUTPUT_PARAMS
noutput=1
tout=0.2
/

&HYDRO_PARAMS
gamma=1.4
courant_factor=0.8
slope_type=1
riemann='hlld'
/
```

Compile RAMSES with directives SOLVER=mhd and NDIM=1

Note B_x=constant in the initial conditions.

Test various Riemann solvers

‘llf’ for Local Lax-Friedrich

‘hll’ for 2-wave (fast magnetosonic) HLL

‘hlld’ for 5-wave HLL-like solver

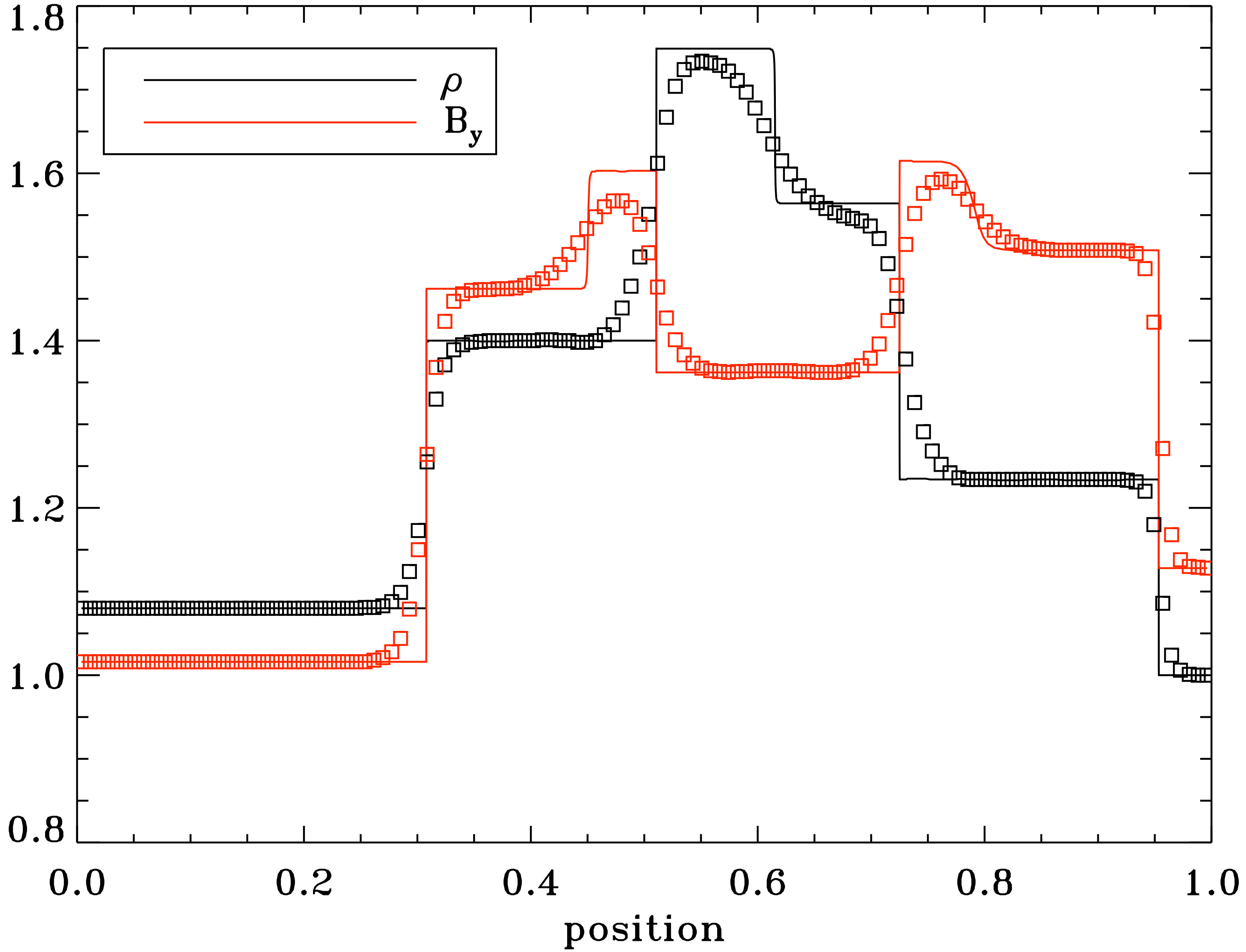
‘roe’ for the 7-wave MHD Roe solver.

Test adapted from Miyoshi & Kusano (2005)

It features a stationary slow magnetosonic shock !

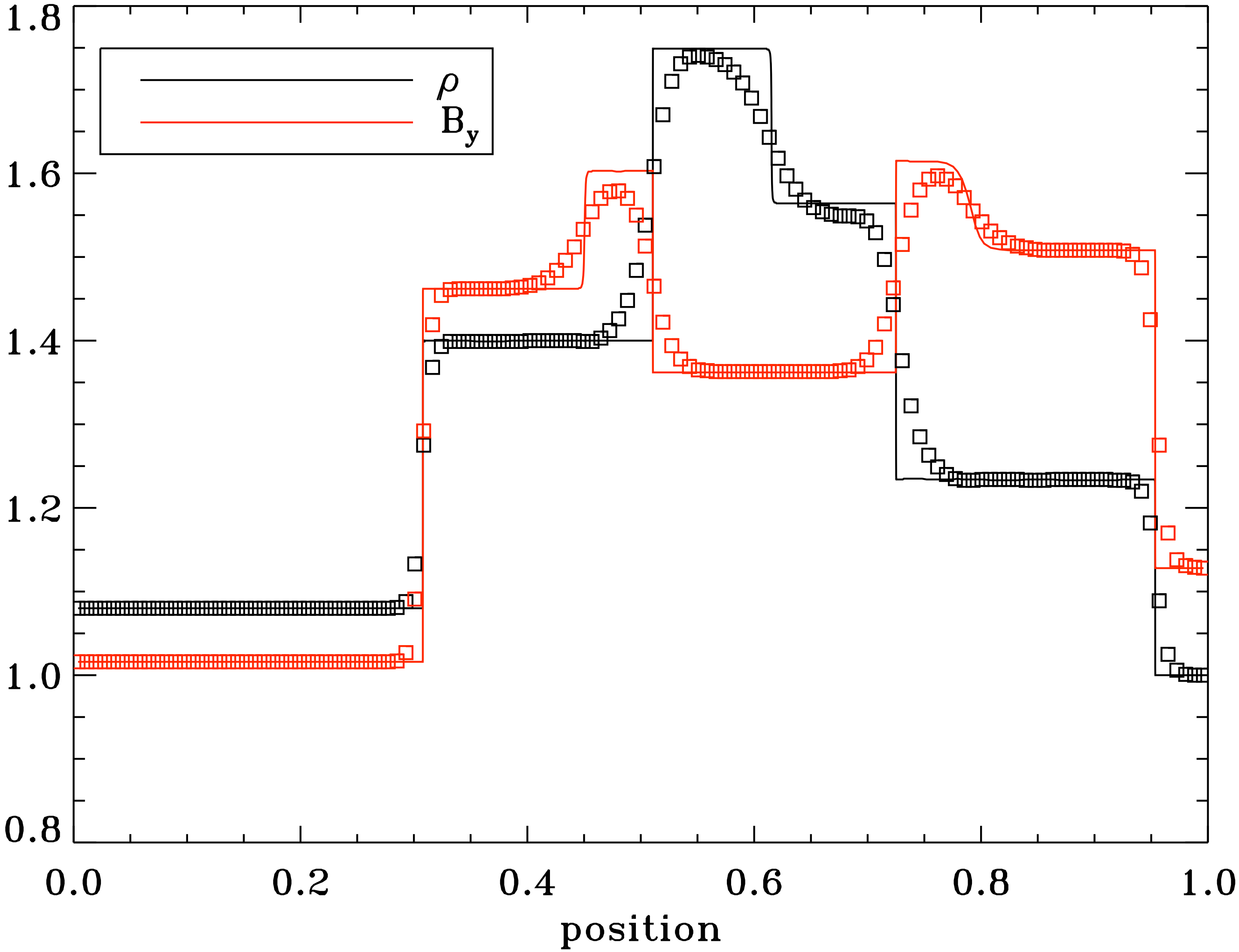
The MHD Lax-Friedrich Riemann solver

```
riemann='llf'
```



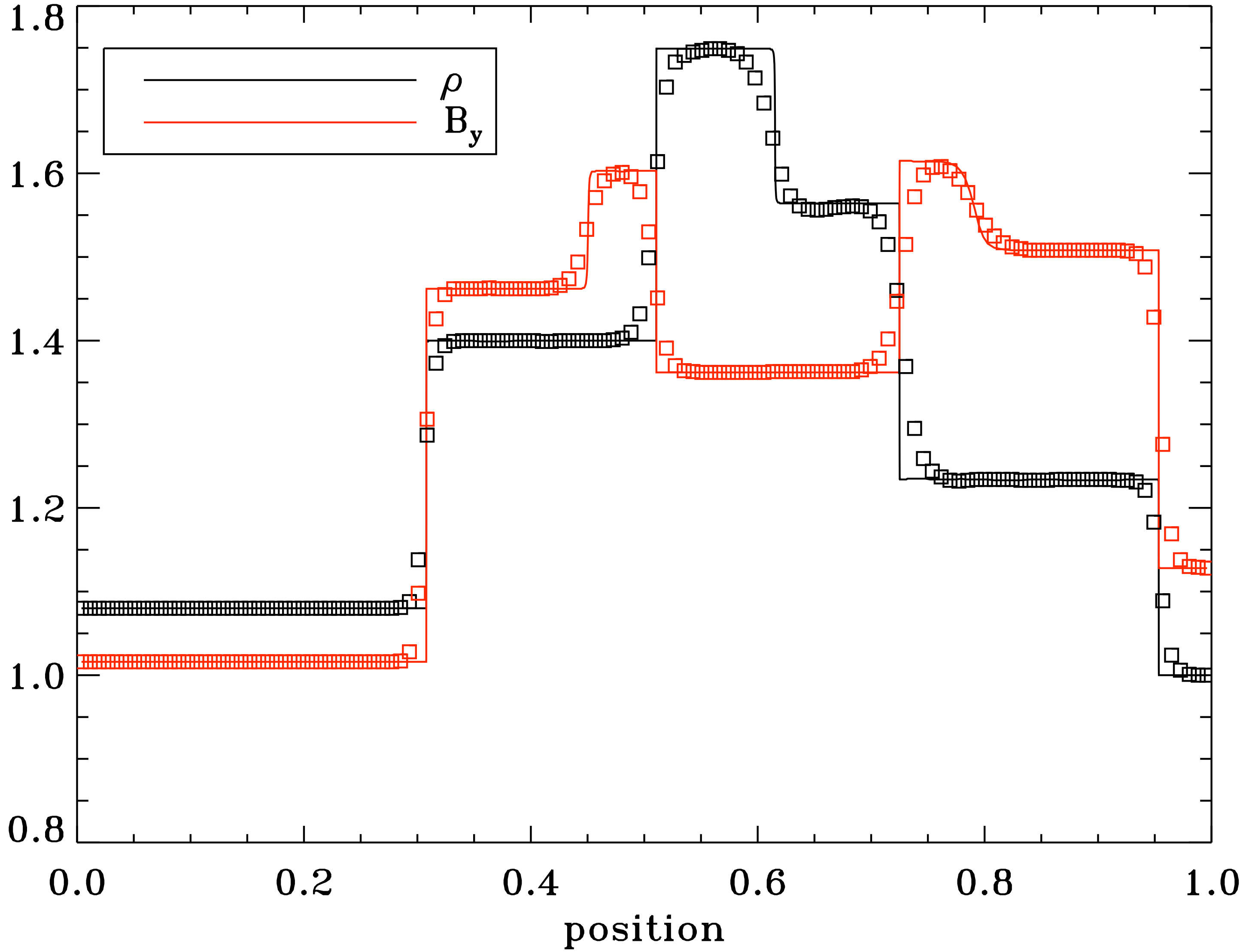
The MHD HLL solver

`riemann='hll'`



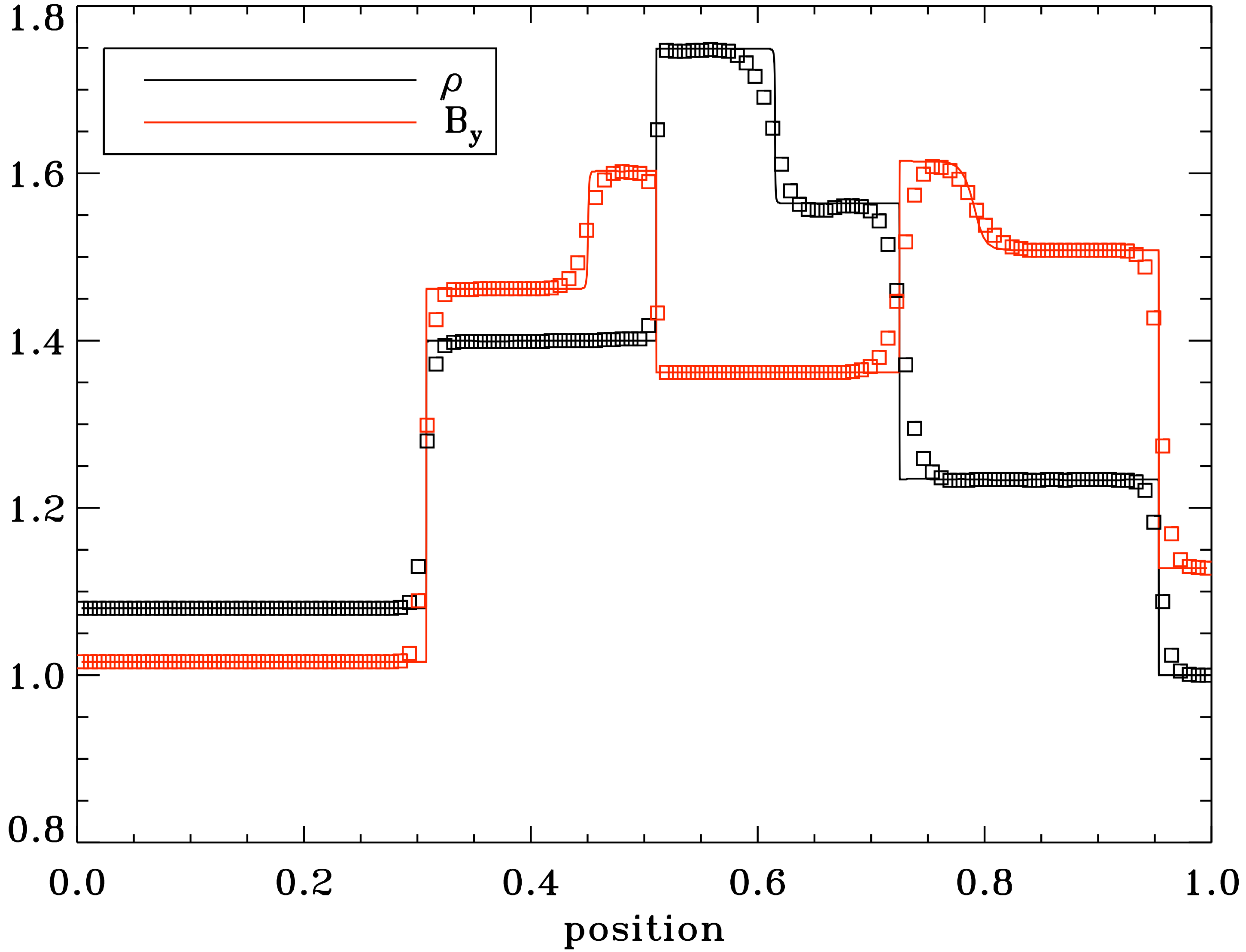
The HLLD solver

```
riemann='hlld'
```



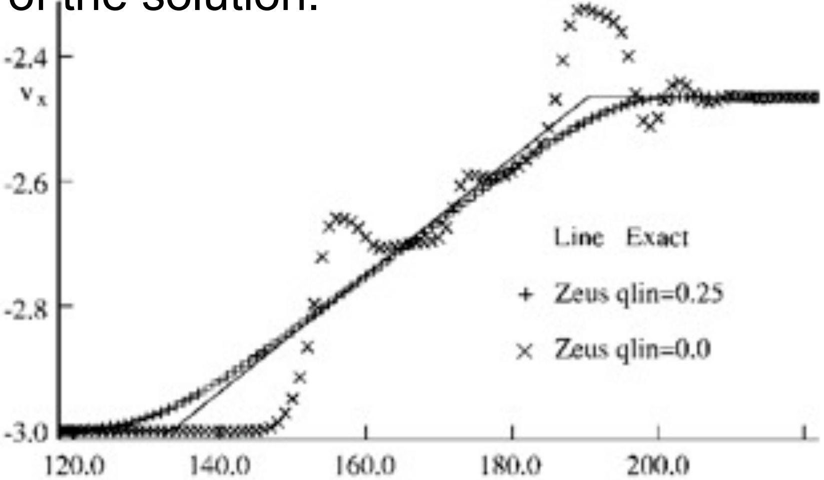
The MHD Roe solver

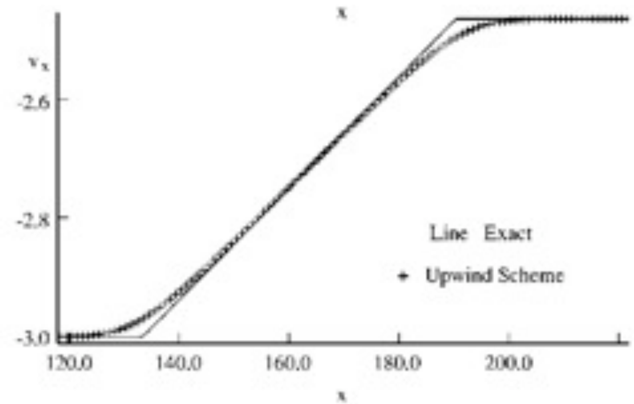
`riemann='roe'`



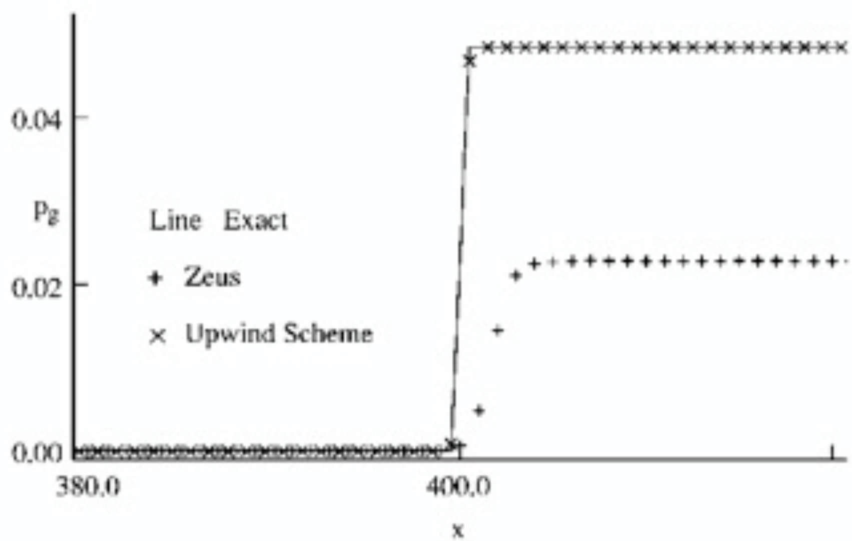
Why bother with a Godunov scheme ?

Proper upwinding of the numerical flux with respect to all 7 waves ensures stability of the solution.





Using a strict conservative update ensures proper jump conditions/shock velocities.



Falle (2000)

Godunov method for MHD in multiple dimensions

The main difficulty is to keep a vanishing divergence for ***B***.

Why is there a problem ?

From the relation: $\nabla \cdot \left(\frac{B^2}{2}\delta_{ij} - B_i B_j \right) + (\nabla \cdot \mathbf{B})\mathbf{B} = -\mathbf{J} \times \mathbf{B}$

We derive the conservative form of the momentum equation with a spurious force:

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} - \frac{1}{4\pi}\mathbf{B}\mathbf{B}) + \nabla P_{tot} = -(\nabla \cdot \mathbf{B})\mathbf{B}$$

On the other hand, we know from Faraday's law $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ that $\partial_t(\nabla \cdot \mathbf{B}) = 0$

If magnetic monopoles are forming due to numerical truncation errors, the induction equation doesn't remove them.

Non-zero divergence accumulates, giving rise to a spurious force parallel to the field lines. In some cases, div B will grow without bounds (numerical instability).

For long time integration, this lead to inconsistent results and quite often to code crashes.

The goal of computational MHD is to design div B preserving schemes.

Cell-centered Godunov method for MHD

Natural extension of finite-volume Godunov schemes to MHD equations.

Define a volume-average magnetic field **B** in a cell *V* as:

$$\mathbf{B}_{ijk} = \frac{1}{V}\int_V \mathbf{B}(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

with $V = [x_{i-1/2}, x_{i+1/2}] \times [y_{i-1/2}, y_{i+1/2}] \times [z_{i-1/2}, z_{i+1/2}]$

Divergence cleaning methods

- Powell's 8-wave scheme (Powell 1999)
- Projection scheme (Brackbil & Barnes 1980)
- Dedner's diffuson scheme (Dedner et al. 2002)

A little bit of everything (Crockett et al. 2005)

div B cleaning schemes

Powell (1999) explicitly introduces magnetic monopole and magnetic current

Add source terms to the momentum equation and to the induction equation

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u}\mathbf{u} - \frac{1}{4\pi}\mathbf{B}\mathbf{B}) + \nabla P_{tot} = -(\nabla \cdot \mathbf{B})\mathbf{B}$$

$$\partial_t \mathbf{B} + c\nabla \times \mathbf{E} = -(\nabla \cdot \mathbf{B})\mathbf{u}$$

← magnetic current

Pros: magnetic monopoles are advected away. Powell's system is still hyperbolic.

Cons: the resulting scheme is not conservative. Jump relations are incorrect.

In 1D, B_x is not constant anymore (it is advected at the flow velocity).

We now have 8 conservative variables with 8 waves (the "div B" wave).

Modify all Riemann solvers to account for this additional degree of freedom.

For $x/t > u^* \quad B_x = B_{x,R}$ and for $x/t < u^* \quad B_x = B_{x,L}$

In particular, at the interface, one get the upwind state for the normal component:

if $u^* > 0 \quad B^{n+1/2}_{x,i+1/2} = B_{x,L}$ if $u^* < 0 \quad B^{n+1/2}_{x,i+1/2} = B_{x,R}$

div B cleaning by the projection method

The previous step gives a normal magnetic flux with non-zero divergence.

Brackbill & Barnes (1980) proposed to remove explicitly magnetic monopoles using the *Projection Method* (also used in incompressible fluids)

Compute the monopole (magnetic charge) for each cell

$$m_{ij} = (B^{n+1/2}_{x,i+1/2,j} - B^{n+1/2}_{x,i-1/2,j})/\Delta x + (B^{n+1/2}_{y,i,j+1/2} - B^{n+1/2}_{y,i,j-1/2})/\Delta y$$

Solve for the potential with the Poisson equation $\Delta\Phi = m$

Correct the normal magnetic field with $\mathbf{B}^{clean} = \mathbf{B}^{n+1/2} - \nabla\Phi$

Use this corrected field in the final conservative update.

It can be shown that this corrected field is the zero-divergence field closest (using the L2 norm) to the original one.

Problems: Poisson equation is non-local (elliptic) and time consuming.

Corrections in the magnetic field can result in large truncation errors in the gas pressure.

Dedner et al. (2002) develops a variant of the scheme, with an hyperbolic div B cleaning step that works also close to stagnation points.

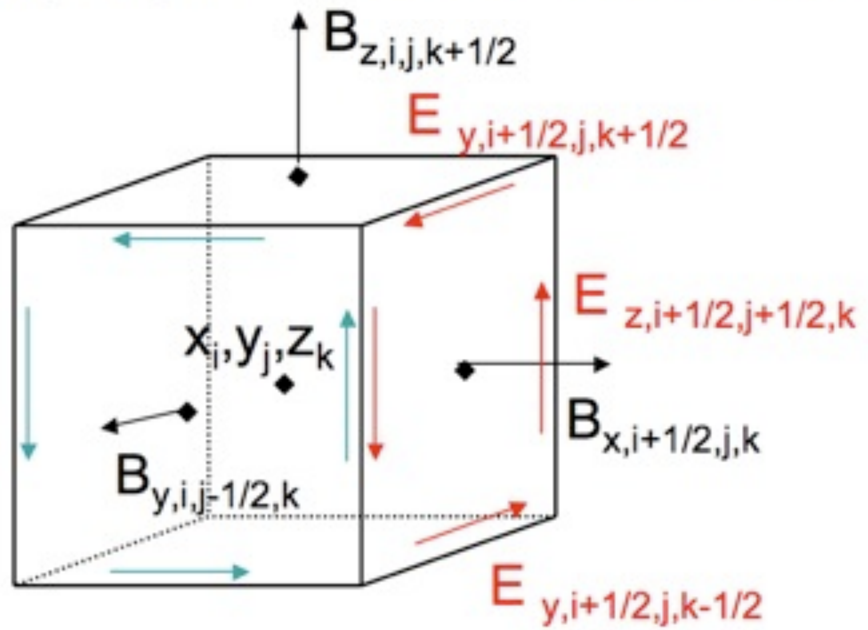
Godunov method with Constrained Transport

The induction equation in integral form suggests a surface-average form:

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0 \qquad \text{(Stokes theorem)} \qquad \partial_t \int_S \mathbf{B} \cdot d\mathbf{s} + \int_L (\mathbf{B} \times \mathbf{u}) \cdot d\mathbf{l} = 0$$

The magnetic field is face-centred while Euler-type variables are cell-centred (staggered mesh approach).

$$(B_x)_{i+1/2,jk} = \frac{1}{S} \int_S B_x(y,z) \mathrm{d}y\mathrm{d}z \qquad S = [y_{i-1/2}, y_{i+1/2}] \times [z_{i-1/2}, z_{i+1/2}]$$



Similar to potential vector methods (Yee 1966; Dorfi 1986; Evans & Hawley 1988).

CT: exact div B preserving scheme

Surface-averaged magnetic fields are updated conservatively:

$$B^{n+1}_{z,i,j,k-1/2} = B^{n}_{x,i,j,k-1/2} + \frac{\Delta t}{\Delta x}\left(E^{n+1/2}_{y,i+1/2,j,k-1/2} - E^{n+1/2}_{y,i-1/2,j,k-1/2}\right) - \frac{\Delta t}{\Delta y}\left(E^{n+1/2}_{x,i,j+1.2,k-1/2} - E^{n+1/2}_{x,i,j-1/2,k-1/2}\right)$$

$$B^{n+1}_{y,i,j-1/2,k} = B^{n}_{y,i,j-1/2,k} + \frac{\Delta t}{\Delta z}\left(E^{n+1/2}_{x,i,j-1/2,k+1/2} - E^{n+1/2}_{x,i,j-1/2,k-1/2}\right) - \frac{\Delta t}{\Delta x}\left(E^{n+1/2}_{z,i+1/2,j-1/2,k} - E^{n+1/2}_{z,i-1/2,j-1/2,k}\right)$$

$$B^{n+1}_{x,i-1/2,j,k} = B^{n}_{x,i-1/2,j,k} + \frac{\Delta t}{\Delta y}\left(E^{n+1/2}_{z,i-1/2,j+1/2,k} - E^{n+1/2}_{z,i-1/2,j-1/2,k}\right) - \frac{\Delta t}{\Delta z}\left(E^{n+1/2}_{y,i-1/2,j,k+1/2} - E^{n+1/2}_{y,i-1/2,j,k-1/2}\right)$$

using time-averaged electric fields defined at cell edge centres:

$$E^{n+1/2}_{x,i,j-1/2,k-1/2} = \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} E_x(x, y_{j-1/2}, z_{k-1/2})\,\mathrm{d}t\,\mathrm{d}x$$

$$E^{n+1/2}_{y,i-1/2,j,k-1/2} = \frac{1}{\Delta t \Delta y}\int_{t^n}^{t^{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}} E_y(x_{i-1/2}, y, z_{k-1/2})\,\mathrm{d}t\,\mathrm{d}y$$

$$E^{n+1/2}_{z,i-1/2,j-1/2,k} = \frac{1}{\Delta t \Delta z}\int_{t^n}^{t^{n+1}}\int_{z_{k-1/2}}^{z_{k+1/2}} E_z(x_{i-1/2}, y_{j-1/2}, z)\,\mathrm{d}t\,\mathrm{d}z$$

The total flux (div B) across each cell bounding surface vanishes exactly.

The induction equation in 2D

We write Faraday's law $\partial_t \mathbf{B} = \nabla \times \mathbf{E}$ using now the EMF vector $\mathbf{E} = \mathbf{u} \times \mathbf{B}$

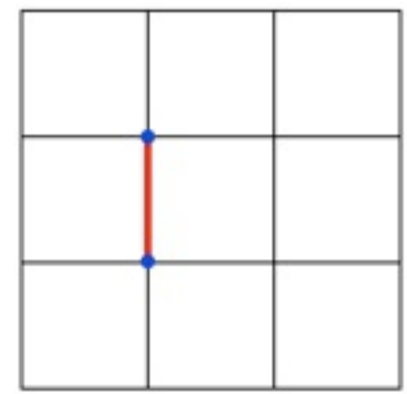
We use a finite-surface approximation for the magnetic field

$$B^n_{x,i+1/2,j} = \frac{1}{\Delta y}\int_{y_{j-1/2}}^{y_{j+1/2}} B_x(x_{i+1/2}, y)\mathrm{d}y \qquad B^n_{y,i,j+1/2} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} B_y(x, y_{j+1/2})\mathrm{d}x$$

Integral form of the induction equation using Stoke's theorem

$$B^{n+1}_{x,i+1/2,j} = B^n_{x,i+1/2,j} + \frac{\Delta t}{\Delta y}\left(E^{n+1/2}_{z,i+1/2,j+1/2} - E^{n+1/2}_{z,i-1/2,j+1/2}\right)$$

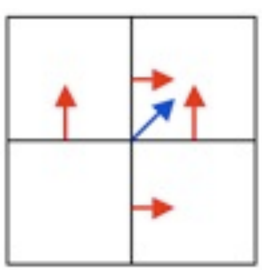
$$B^{n+1}_{y,i,j+1/2} = B^n_{y,i,j+1/2} + \frac{\Delta t}{\Delta x}\left(E^{n+1/2}_{z,i+1/2,j+1/2} - E^{n+1/2}_{z,i+1/2,j-1/2}\right)$$



By construction, div B vanishes exactly:

$$\frac{B^n_{x,i+1/2,j} - B^n_{x,i-1/2,j}}{\Delta x} + \frac{B^n_{x,i,j+1/2} - B^n_{x,i,j-1/2}}{\Delta y} = 0$$

For piece-wise initial constant data, the flux function is self-similar at corner points.



Induction Riemann problem

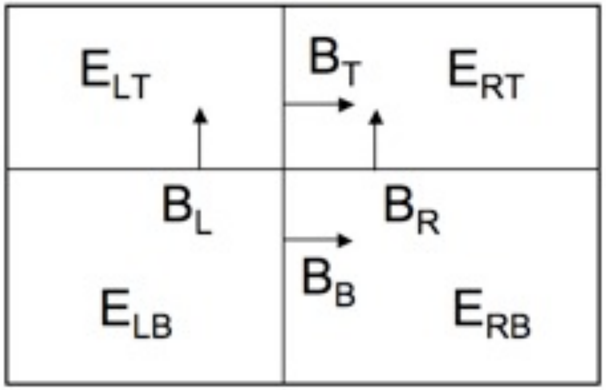
For pure induction, the exact Riemann solution is:

$$E^{n+1/2}_{z,i+1/2,j+1/2} = u\frac{B^n_{y,i+1,j+1/2} + B^n_{y,i,j+1/2}}{2} - v\frac{B^n_{x,i+1/2,j+1} + B^n_{y,i+1/2,j}}{2}$$

$$- |u|\frac{B^n_{y,i+1,j+1/2} - B^n_{y,i,j+1/2}}{2} + |v|\frac{B^n_{x,i+1/2,j+1} - B^n_{y,i+1/2,j}}{2}$$

2D Riemann solvers for MHD

Londrillo & Del Zana 2004, Gardiner & Stone 2005, Teyssier et al. 2006; Fromang et al. 2006, Balsara 2010



1- Linear 2D Riemann solvers:

For a 1D Riemann solver (e.g. Roe):

$$F(U(0)) = \frac{1}{2}(F_L + F_R) + \frac{1}{2}\sum_{i=1,m}|\tilde{\lambda}_i|(\tilde{\beta}_i - \tilde{\alpha}_i)\tilde{K}^i$$

The 2D flux is given by:

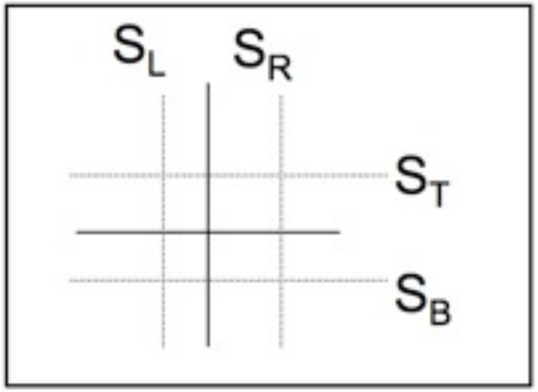
$$F(U(0)) = \frac{1}{4}(F_{LL} + F_{RL} + F_{LR} + F_{RR}) + \frac{1}{2}\sum_{i=1,m}|\tilde{\tilde{\lambda}}_{x,i}|(\tilde{\tilde{\beta}}_{x,i} - \tilde{\tilde{\alpha}}_{x,i})\tilde{\tilde{K}}_x^i - \frac{1}{2}\sum_{i=1,m}|\tilde{\tilde{\lambda}}_{y,i}|(\tilde{\tilde{\beta}}_{y,i} - \tilde{\tilde{\alpha}}_{y,i})\tilde{\tilde{K}}_y^i$$

`riemann2d='llf'`

`riemann2d='roe'`

2- The HLL solver in 2D: `riemann2d='hll'`

$$E^* = \frac{S_R S_T E_{LB} + S_L S_B E_{RT} - S_L S_T E_{RB} - S_R S_B E_{LT}}{(S_R - S_L)(S_T - S_B)} - \frac{S_B S_T}{S_T - S_B}(B_R - B_L) - \frac{S_L S_R}{S_R - S_L}(B_T - B_B)$$

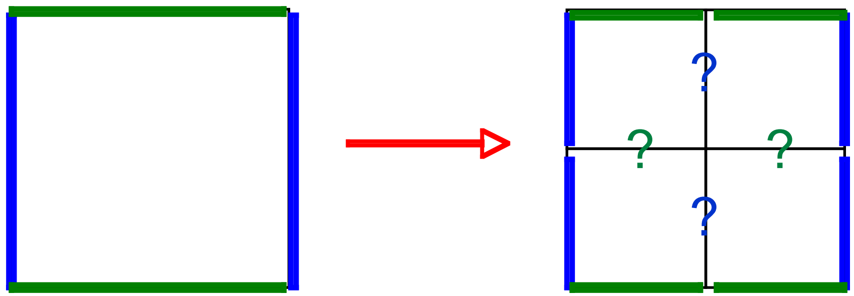


3- The HLLD solver in 2D: `riemann2d='hlld'`

Higher-order schemes and AMR

« Divergence-free preserving » restriction and prolongation operators

Balsara (2001) Toth & Roe (2002)



Flux conserving interpolation and averaging within cell faces using TVD slopes in 2 dimensions

EMF correction for conservative update at coarse-fine boundaries

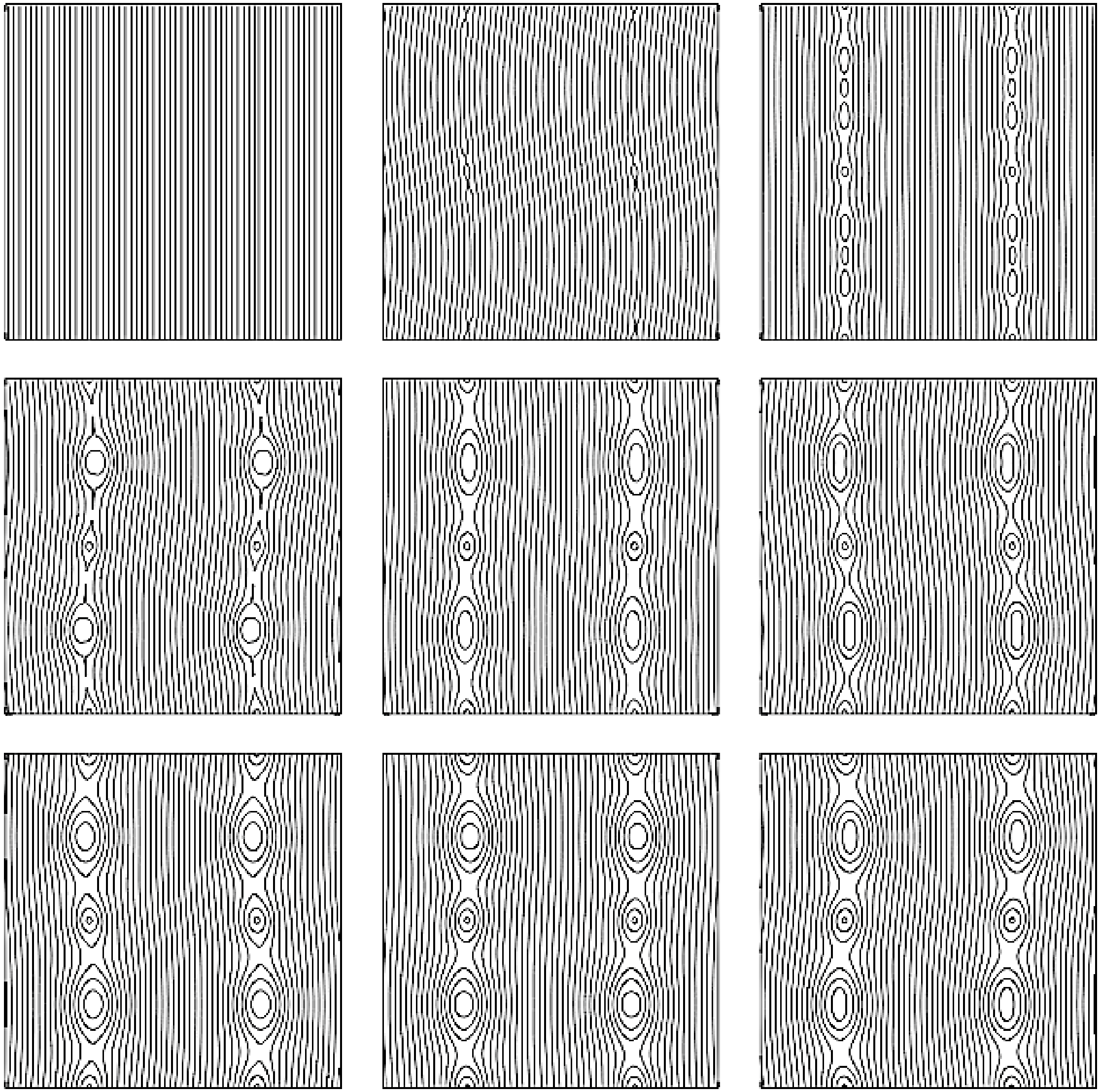
$$\langle E_z \rangle_k^\ell \Delta z_\ell \Delta t_\ell^n = \langle E_z \rangle_{2k}^{\ell+1} \Delta z_{\ell+1} \Delta t_{\ell+1}^{2n} + \langle E_z \rangle_{2k+1}^{\ell+1} \Delta z_{\ell+1} \Delta t_{\ell+1}^{2n+1}$$

For a fully second-order MUSCL scheme for MHD, see Teyssier et al. 2006; Fromang et al. 2006.

In particular, the predictor step requires MHD states to be Taylor expanded at cell faces but also at cell corners.

The current sheet test

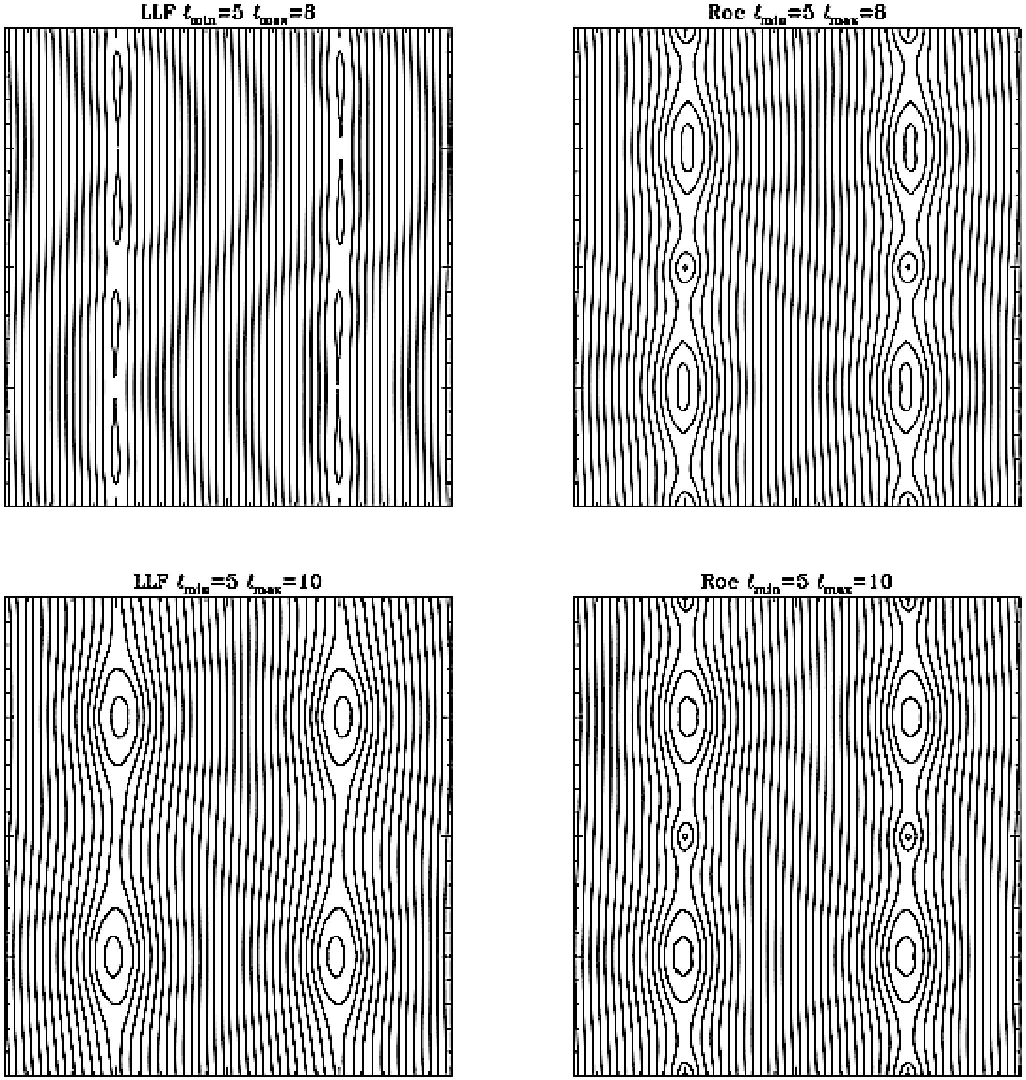
Magnetic reconnection occurs because of numerical diffusion



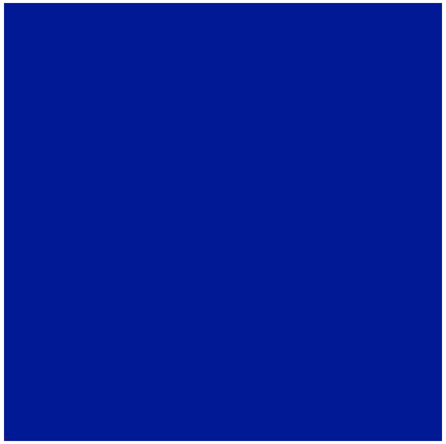
Time sequence of field lines (Fromang et al. 2006)

The current sheet test

Different Riemann solvers converge towards different solutions !



Collapse of a molecular core and star formation

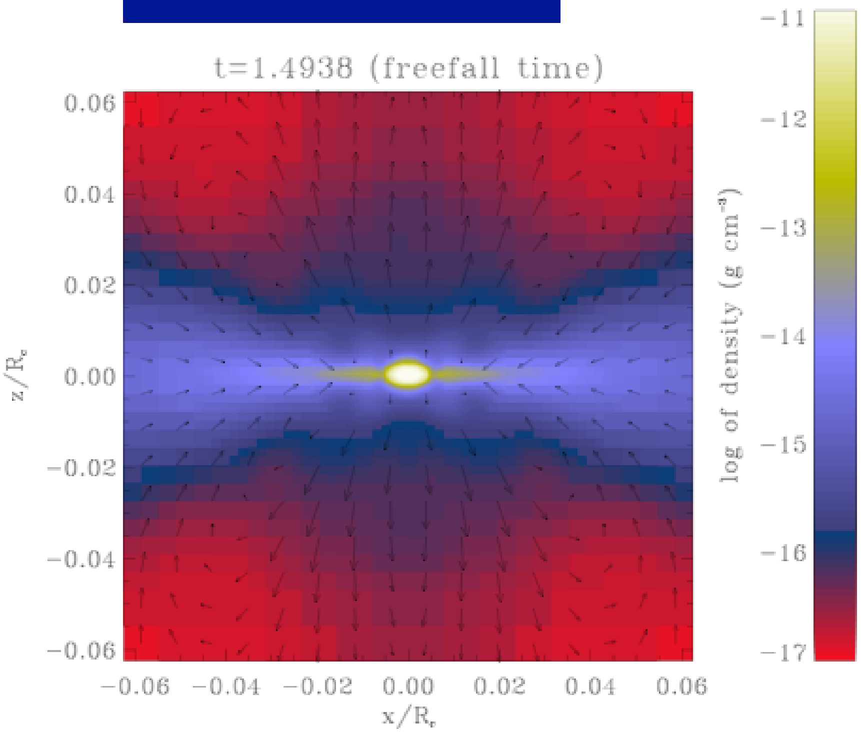


2 main effects of magnetic fields on the collapse:

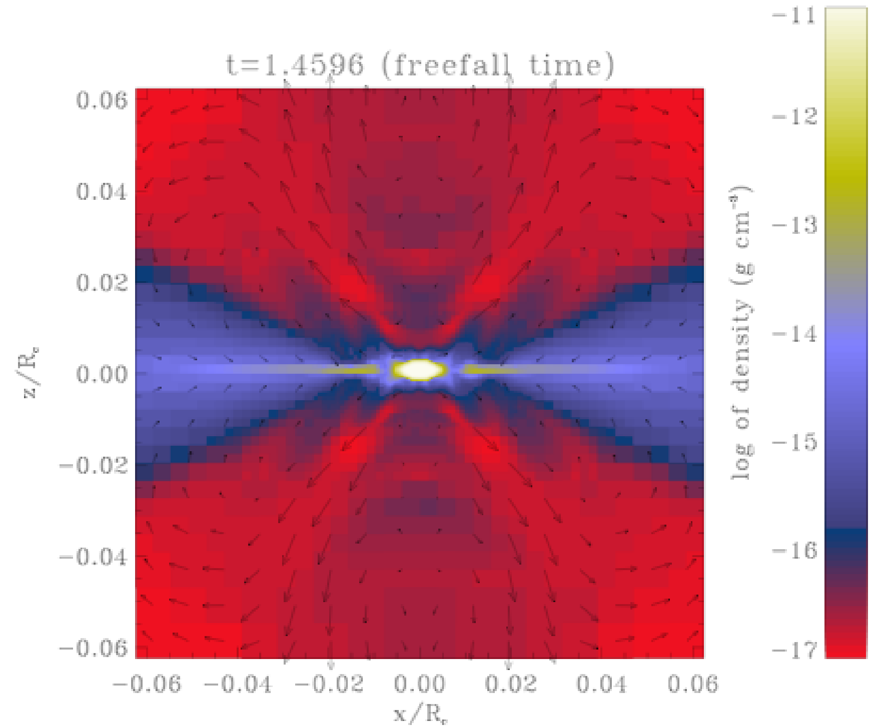
- magnetic breaking (remove angular momentum in escaping Alfven waves)

- build-up of a magnetic tower and launch of a conical jet

Fromang et al. 2006: Hennebelle & Teyssier 2007



LLF Rieman solver



Roe Riemann solver

Conclusion

- Ideal MHD equations in 1D can be modeled using fully cell-centered Godunov schemes

- We have designed several MHD Riemann solvers (they are all present in RAMSES)

- In 2D and 3D, MHD equations are more problematic: numerical build-up of magnetic monopoles: instabilities and spurious forces

- Fully cell-centered schemes can be developed using div B cleaning (non-conservative, non-local)

- Face-centered schemes are more natural (exact magnetic flux conservation and vanishing divergence are easy to obtain)

- Constrained Transport approach requires proper 2D upwinding of MHD waves to compute the electric field: 2D Riemann solvers and 2D slope limiters.

References

Balsara, D., 2010, JCP, 229, 6
Brackbill, J., Barnes, D., 1980, JCP, 35, 426
Brio, M., Wu, C., 1988, JCP, 75, 400
Cargo, P., Gallice, G., 1997, JCP, 136, 446
Crockett, R., Collela, P., Fisher, R., Klein, R., McKee, C., JCP, 203, 422
Dai, W., Woodwards, P., 1998, ApJ, 494, 317
Dedner, A., Kemm, F., Kroner, D., et al. 2002, JCP, 175, 645
Dorfi, E., JCP, 43, 1
Evans, C., Hawley, J., 1988, ApJ, 33, 659
Fromang, S., Hennebelle, P., Teyssier, R., 2006, A&A, 457, 371
Gardiner, T., Stone, J., 2005, JCP, 205, 509
Gurski, K, 2004, SIAM J., SCI Comput, 25, 2165
Londrillo, P., Del Zanna, L., 2000, JCP, 195, 17
Miyoshi, T., Kusano, K, 2005, JCP, 208, 315
Powell, K., Roe, P., Linde T. et al., 1999, JCP, 154, 284
Roe, P., 1981, JCP, 53, 357
Ryu, D., Miniati, F., Jones, T., Franck, A., 1998, ApJ, 509, 244
Stone, J., Norman, M., 1992, ApJS, 80, 791
Teyssier, R., Dormy, E. Fromang, S., 2006, JCP, 218, 44
Toth, G., 2000, JCP, 161, 605