References: 0

"Numerical Methods for Conservation Law"
R. LeVeque (Birkhauser)

"Computational Methods for Astrophysical Flow"
LeVeque (Springer)

"Computional Gas Dynamics"
C. Laney (Cambridge)

Andrew MacFadyen (NYU)
macfadyen@nyu.edu

1. Intro.

Hyperbolic System

Conservation Laws

Time dependent, non-linear PDEs.

Simplicity 1D

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial f}{\partial x}(u(x,t)) = 0$$

$$u_t + f_x = 0$$

u is an m-dimensional vector of state variables or densities of conserved quantities.
e.g. mass, momentum, energy

$f(u)$ is the flux function

where f_j is the flux of

s.v. j.

For Euler eqns. $m=3$

$m=1$ mass ρ

$m=2$ momentum ρv

$m=3$ energy E (total here)

$[x_1, x_2]$

$$\int_{x_1}^{x_2} u_j(x,t)\,dx$$ is total s.v. in $[x_1, x_2]$ at time t

e.g. $m=1$ $u_1 = \rho$, mass

Initial Value Problem (IVP)

must specify initial condition

" " boundary " "

$u(x,0) = u_0(x)$

solve for $u(x,t)$ for $t \geq 0$

Hyperbolic if Jacobian Matrix $f'(u)$

1. is diagonalizable $\Rightarrow$ complete set of linearly independent eigenvectors.
2. eigenvalues are real

In 3D

$$\frac{\partial}{\partial t} u(x,y,z,t) + \frac{\partial}{\partial x} F(u(x,y,z,t)) + \frac{\partial}{\partial y} G(u(x,y,z,t)) + \frac{\partial}{\partial z} H(u(x,y,z,t)) = 0$$

$$u_t + f(u)_x + G(u)_y + H(u)_z = 0$$

the flux functions $F(u)$ are non-linear functions of u

Non-linear system of PDE's
$\Rightarrow$ need numerical solutions.
shock formation

Fluids obey Navier-Stokes
Not hyperbolic $\quad f \neq f(u)$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{\rho} \vec{\nabla} P + \nu \nabla^2 v$$

In astrophysics large

$$Re = \frac{\text{inertial}}{\text{viscous}} = \frac{\frac{v^2}{L}}{\nu \frac{v}{L^2}} = \frac{vL}{\nu}$$

$\Rightarrow$ Euler Equations

ignore diffusivities

e.g. viscosity, heat conduction

$$\frac{\partial u}{\partial t} + \nabla \cdot f = 0$$

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho v \\ \rho v^2 + P \\ Ev + Pv \end{bmatrix} = 0$$

u $\qquad$ $\boxed{f(u)}$

EOS $\quad P(\rho, E_t)$ $\quad P = P(u)$

$$u_t + f(u)_x = 0$$

$u_1 = \rho$

$u_2 = M = \rho v$

$u_3 = E$

$f_1 = \rho v$

$f_2 = \rho v^2 + P$

$f_3 = v(E + P)$

$\rho(x,t)$ is mass density

$v(x,t)$ velocity

$M(x,t)$ momentum density ρv

$E(x,t) = E_t + \frac{1}{2}\rho v^2$

$v = \frac{u_2}{u_1}$ $\quad E_* = E - \frac{1}{2}\rho v^2 = u_3 - \frac{1}{2}u_1\left(\frac{u_3}{u_1}\right)^2$

Use Equation of state (EOS)

$$P = E_*(\Gamma - 1) \equiv \rho\epsilon(\Gamma - 1)$$

$E_* = \rho\epsilon$ $\quad \epsilon \equiv$ specific internal energy (erg/gm)

$\Gamma \equiv$ adiabatic index $= \frac{\partial \ln P}{\partial \ln \rho} = \frac{dP/P}{d\rho/\rho}$

non-relativistic $\Gamma = 5/3$

relativistic $\Gamma = 4/3$ $\quad 4/3 < \Gamma < 5/3$ $\quad \Gamma = \Gamma(\epsilon)$

massive star

$$P = \frac{1}{3} a T^4 + \frac{R_g}{\mu} \rho T$$

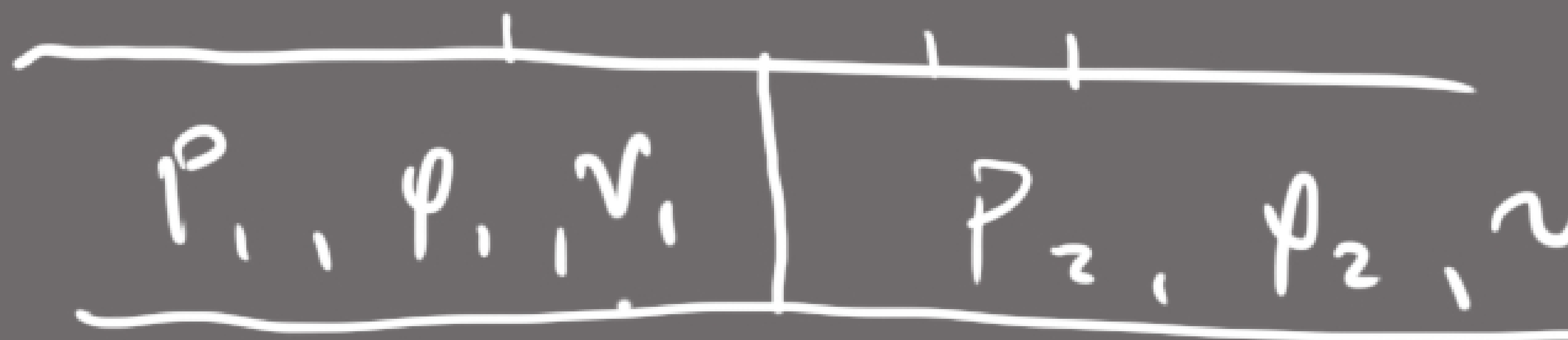
for $T \gtrsim 10^9$

relativistic particles dominate pressure.

But rest mass dominates energy density.

Shock Tube - Test Problem

$P_1 > P_2 \qquad v = 0$



Example. Riemann Problem

2 constant states

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \qquad \frac{\partial P}{\partial x} < 0 \Rightarrow$$

motion to right

get 3 waves:

right 1. shock wave: all s.v. discontinuous

ρ, P increase

moves from high to lo P

right 2. contact discontinuity (C.D.):

density discontinuous

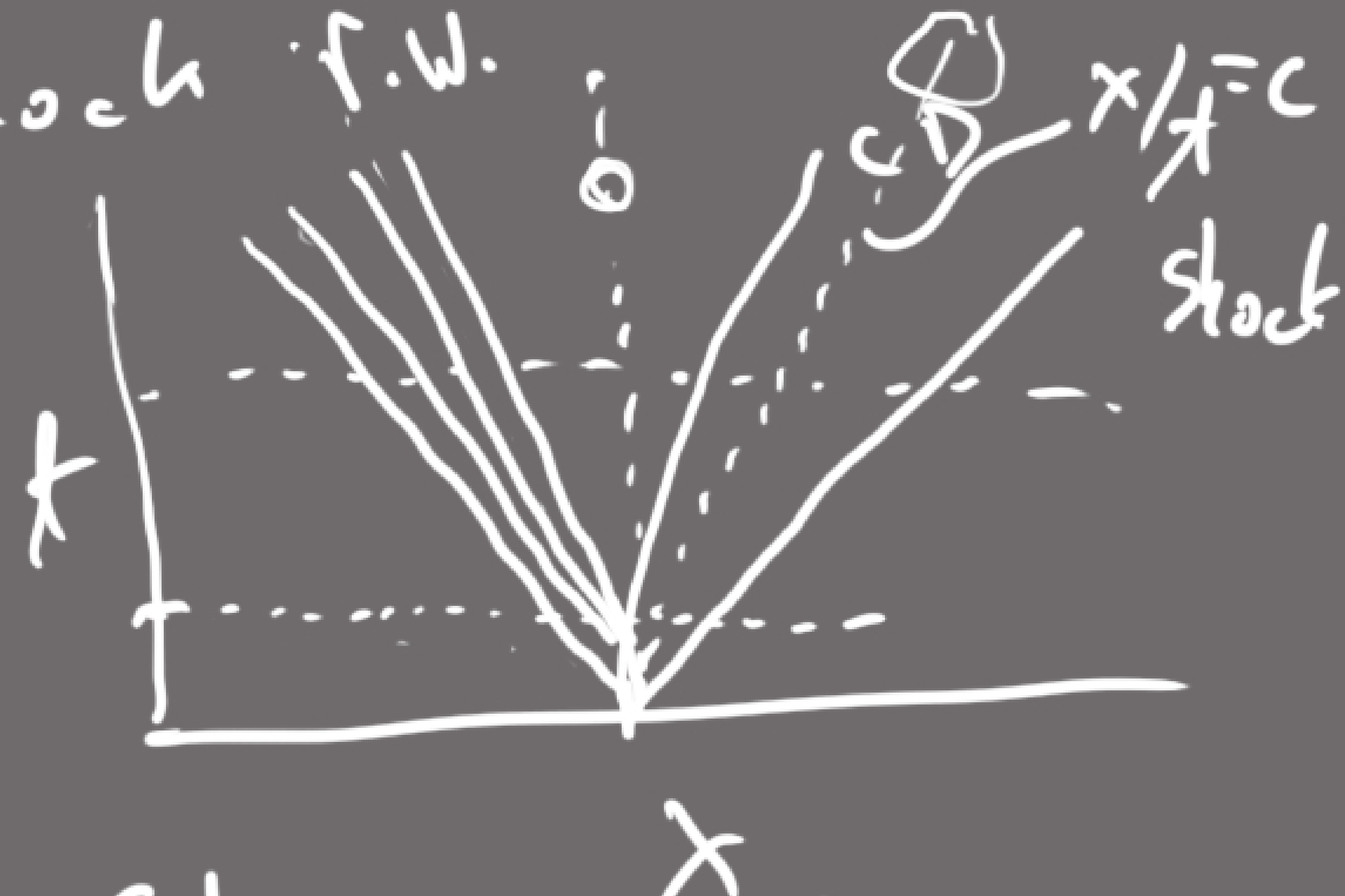
left 3. rarefaction wave (fan)

P, v continuous

all s.v. continuous

smooth transition

density decreases as wave passes

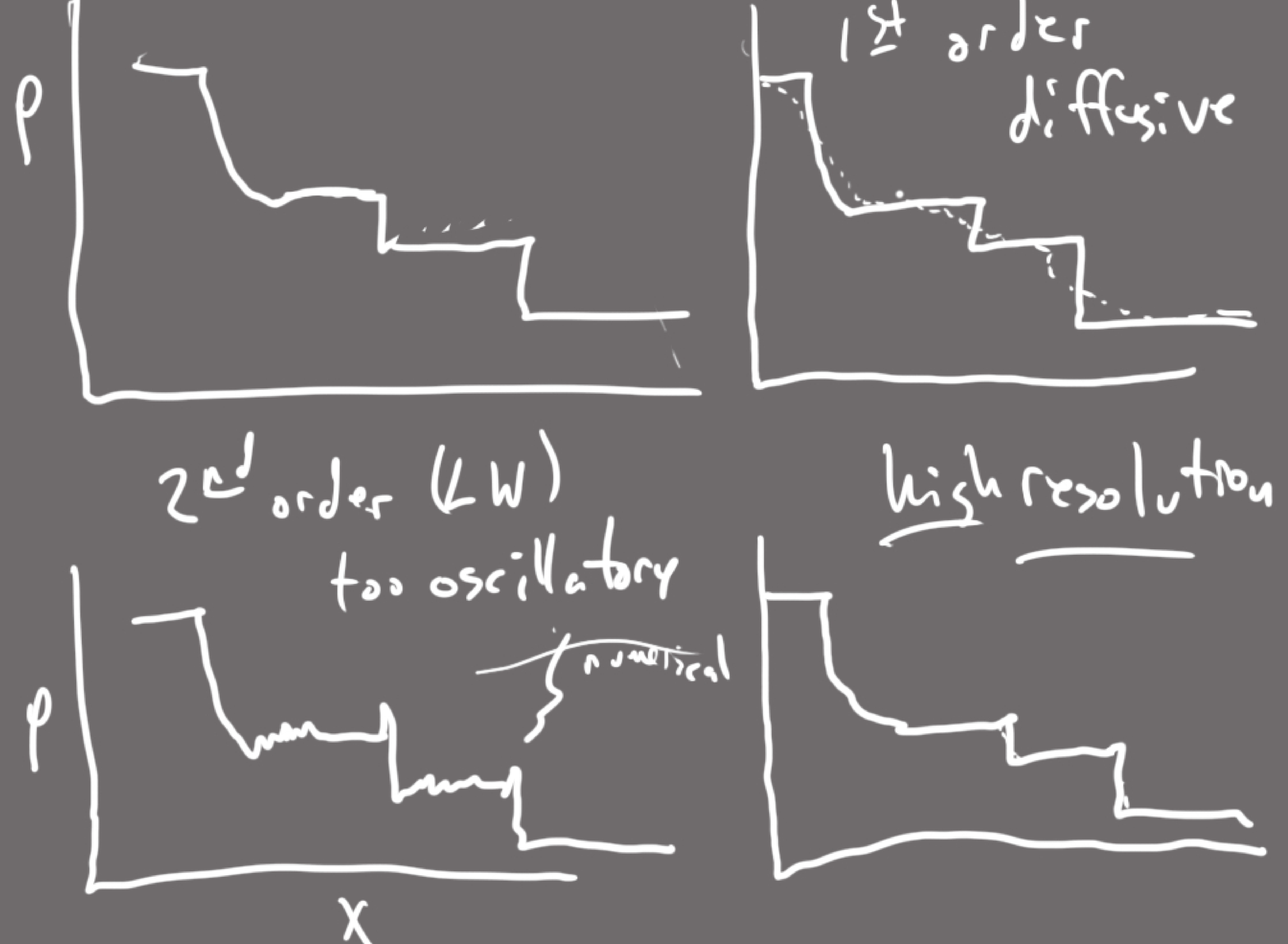


Soln $u(x,t)$ is a similarity solution in variable x/t

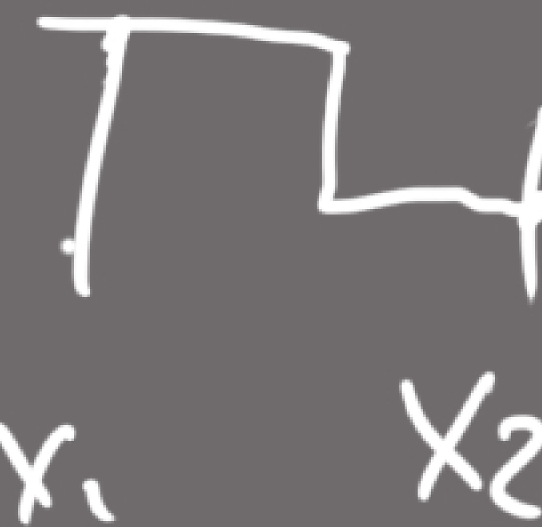
$$u = f(x/t)$$

$$u(x,t) = u(\alpha x, \alpha t)$$

$$u(x,t) = w\left(\frac{x}{t}\right) = w\left(\frac{\alpha x}{\alpha t}\right)$$


ρ
1st order
diffusive
2nd order (LW)
too oscillatory
numerical
ρ
x
high resolution

Conservation Laws



Mass Cons. in 1D

LOCAL cons. of mass

mass in section $[x_1, x_2]$ can only change by gas flowing across x_1 or x_2

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)\cdot v(x_2,t)$$

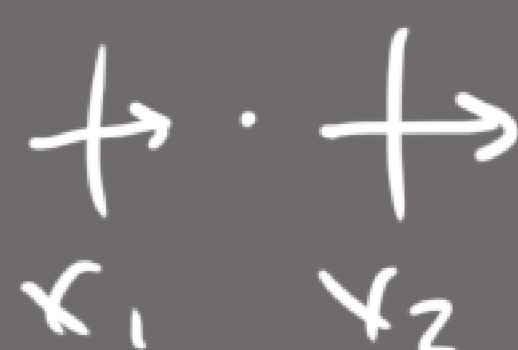
$$\int_{x_1}^{x_2} \rho(x, t_2)\,dx = \int_{x_1}^{x_2} \rho(x, t_1)\,dx$$

$$+ \int_{t_1}^{t_2} \rho(x_1, t)\,v(x_1, t)\,dt$$

mass entering x_1

$$- \int_{t_1}^{t_2} \rho(x_2, t)\,v(x_2, t)\,dt$$

mass leaving x_2



cons. of mass

$$\rho_t + (\rho v)_x = 0$$

Simple case $v = a = \text{const}$

$$\rho v = f(\rho)$$

Scalar cons. law for ρ

$f(\rho) = a\rho$ $\qquad \underline{\rho_t + a\rho_x = 0}$

Linear Scalar Advection Eqn.

for initial data $\rho(x, 0) = \rho_0(x)$

exact solution is $\rho(x,t) = \rho_0(x - at)$

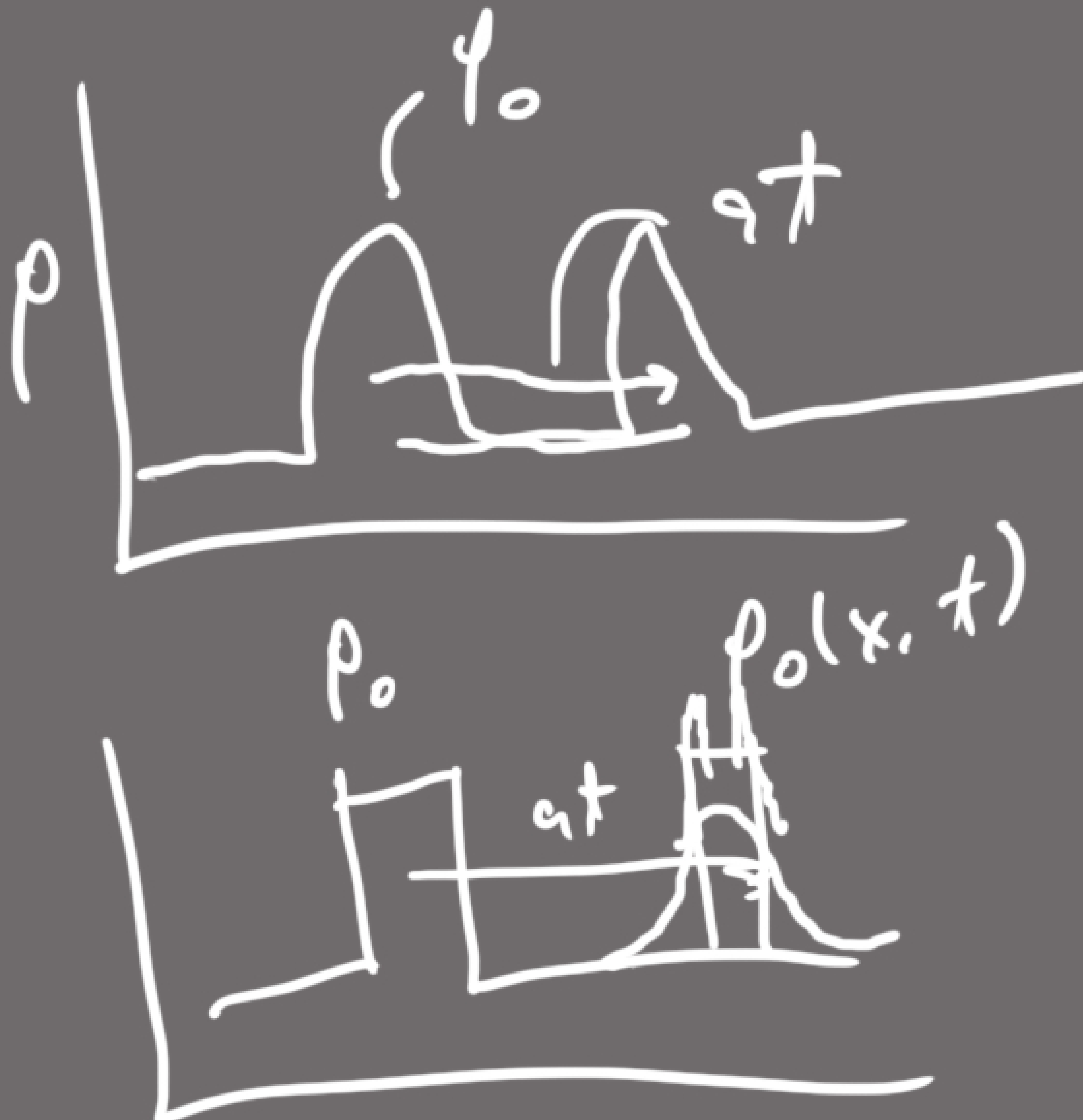
$\rho_t = -a\rho_0 \qquad \rho_x = \rho$

$$\rho_t + a\rho_x = 0$$

$$-a\rho_0 + a\rho = 0$$

initial profile just moves (adjusts) with const. velocity a. no change in shape

⇒ True even if ρ_0 is not smooth



Diffusion (of heat or chemical composition)

"Fouriers Law"

diffusive flux $= -D\rho_x$

from hi to lo

$D>0$

combine w/ advective flux

$f(\rho, \rho_x) = a\rho - D\rho_x$ Parabolic eqn.

$$\rho_t + (q\rho - D\rho_x)_x = 0$$

$$\boxed{\rho_t + q\rho_x = \boxed{D\rho_{xx}} \\ = 0}$$

Sometimes numerical

Vector Matrix

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}}{\partial x} = 0$$

$$\vec{f} = \vec{f}(\vec{u})$$

$$\frac{\partial \vec{f}}{\partial x} = \frac{\partial \vec{f}}{\partial \vec{u}} \frac{\partial \vec{u}}{\partial x} = \overleftrightarrow{A} \frac{\partial \vec{u}}{\partial x}$$

Jacobian Matrix

$f'(u)$

$$\frac{\partial \vec{f}}{\partial \vec{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{bmatrix}$$

$$u = (\underset{u_1}{\rho}, \underset{u_2}{\rho v}, \underset{u_3}{E})$$

$$f = (\rho v, \rho v^2 + P, v(E+P))$$

$$u_2, \quad \frac{u_2^2}{u_1} + P, \quad u_2(u_3+P)/u_1$$

$$f'(u) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\Gamma-3)v^2 & (3-\Gamma)v & (\Gamma-1) \\ \frac{1}{2}(\Gamma-1)v^3 - v(E+P)/\rho & \frac{(E+P)}{\rho} - (\Gamma-1)v^2 & \Gamma v \end{bmatrix}$$

Eigenvalues are

$$\lambda_1(u) = v - c_s \qquad \lambda_2(u) = v \qquad \lambda_3(u) = v + c_s$$

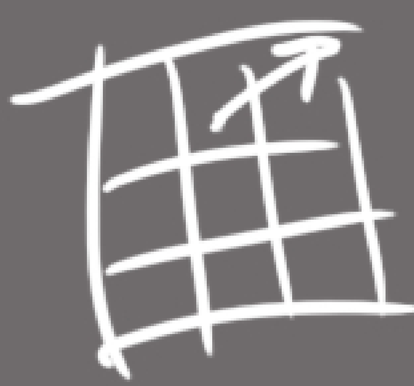
$$c_s = \sqrt{\frac{\Gamma P}{\rho}} = \text{sound speed}$$

Relativistic Hydrodynamics (SRHD)

conservation of rest energy-momentum

$$(\rho u^\mu)_{;\mu} = 0 \qquad \left(T^{\mu\nu}\right)_{;\nu} = 0$$

ρ = rest mass density in fluid (comoving) frame



$$u^\mu = \gamma\left(c, \vec{u}\right)$$

Lorentz factor, $\gamma = \dfrac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$ $\qquad \vec{u}\cdot\vec{u}$

Perfect Fluid (no diffusion)

$$T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu}$$

$$g^{\mu\nu} = \eta^{\mu\nu}$$

Minkowski

$$h = 1 + \epsilon + P/\rho$$

relativistic specific enthalpy

use units for which $c = 1$

$$\rho h = \underbrace{\rho + \rho\epsilon}_{\text{total energy}} + P = \tilde{\rho} + P$$

$g^{\mu\nu}$ spacetime metric T^{0i}

in 1D

$\underline{U_t + f(U)_x = 0}$

$$U = \begin{pmatrix} D \\ S \\ \tau \end{pmatrix} \qquad F = \begin{pmatrix} Dv \\ Sv + P \\ S - Dv \end{pmatrix}$$

$D = \gamma\rho$

$S^j = \rho h \gamma^2 v^j$

$\tau = \rho h \gamma^2 - P - \gamma\rho$

$U = (D, S, \tau)$ conserved vars.

$P = (\rho, v, P)$ primitive vars.

need a Newton-Raphson to recover primitive vars. from conserved vars.