The Linear Universe

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Modelling the Universe

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metric	fluid densities	fluid velocities

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- The laws: Einstein and continuity equations

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Set of coupled PDE's -- need initial conditions!

The Linear Universe

- For much of the history of the Universe:

$$g_{\mu\nu}(x,t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x,t)$$

$$\rho_i(x,t) = \bar{\rho}_i(t) + \delta\rho_i(x,t) \qquad \delta = \text{ small}$$

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Universe

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Homogeneous Background

$$\bar{G}_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu}$$

Fluctuations

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}$$

Initial Conditions

- The homogeneous Universe:

$$p_i = w_i \rho_i \qquad\qquad \bar{\rho}_i(t_0)$$

Types of fluids — Densities today

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Types of fluids — Densities today

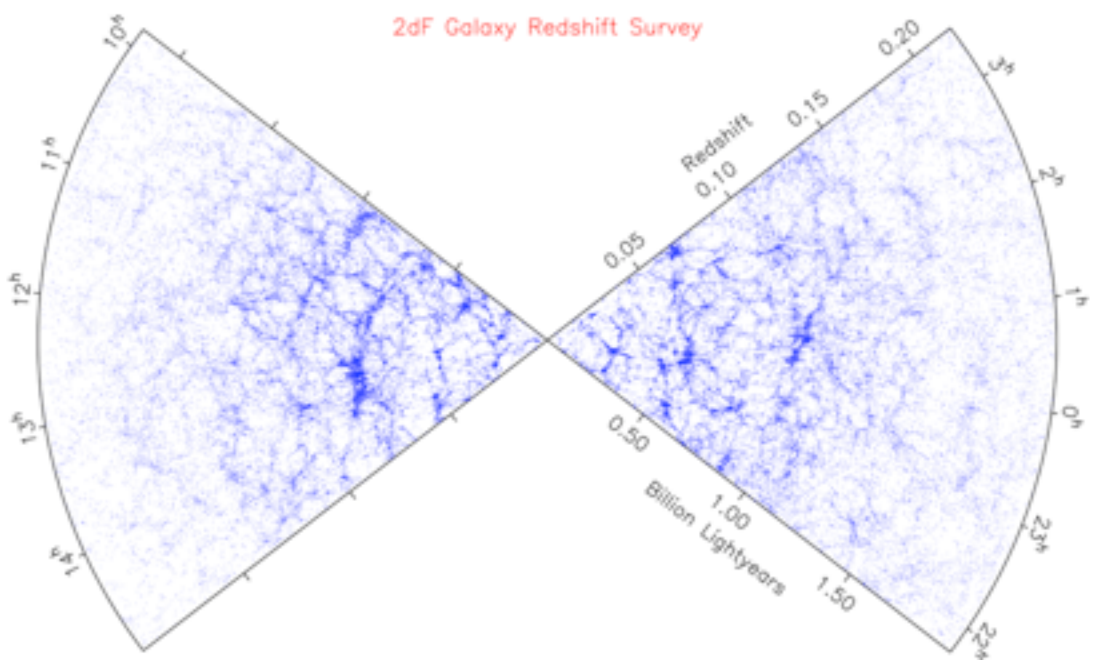
- The Linear Universe:

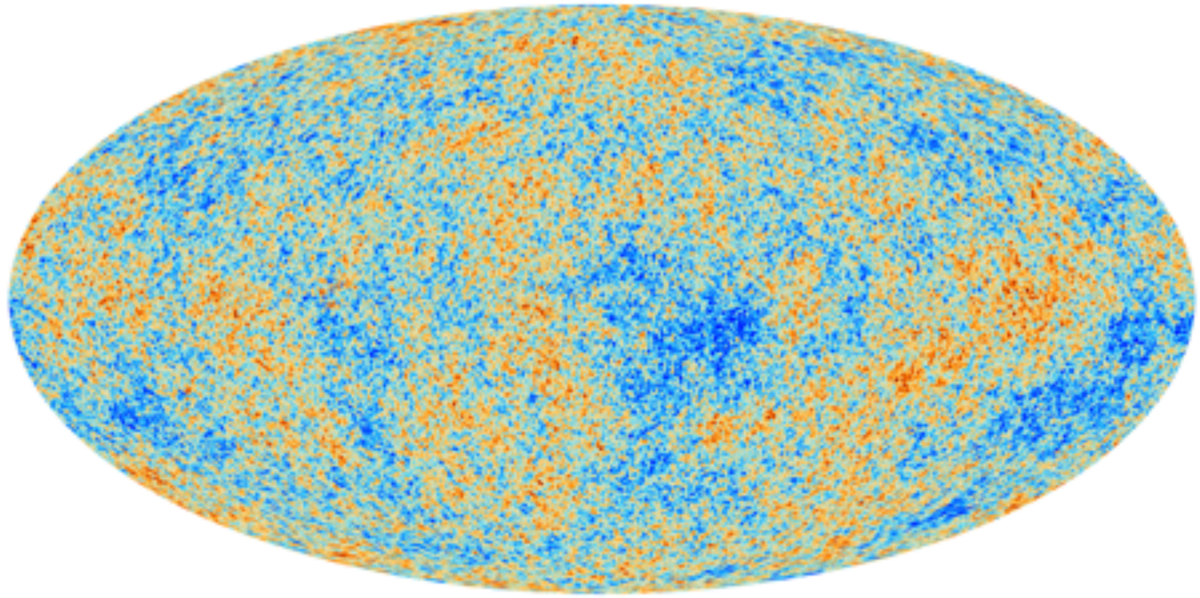
Characterize statistics of inhomogeneities!

$$\mathcal{P}\left[g_{\mu\nu}\left(x, t=0\right)\right] \quad \mathcal{P}\left[\rho_i\left(x, t=0\right)\right] \quad \mathcal{P}\left[u_{i\,\mu}\left(x, t=0\right)\right]$$

Assume our Universe is typical.

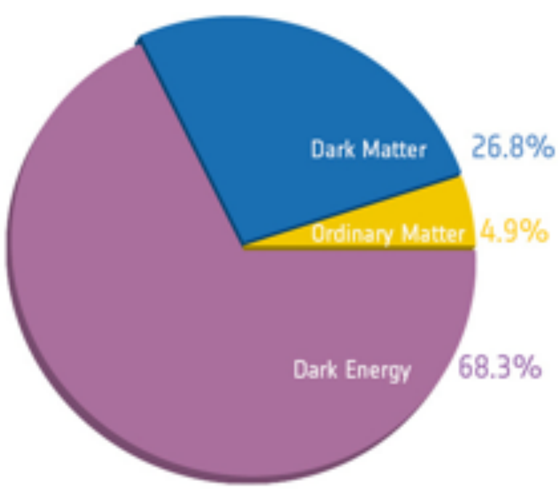
6 Parameter Model of the Universe



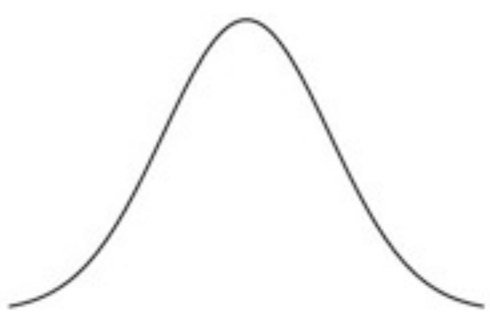


$$\Lambda\mathrm{CDM}$$

3 parameters



2 parameters



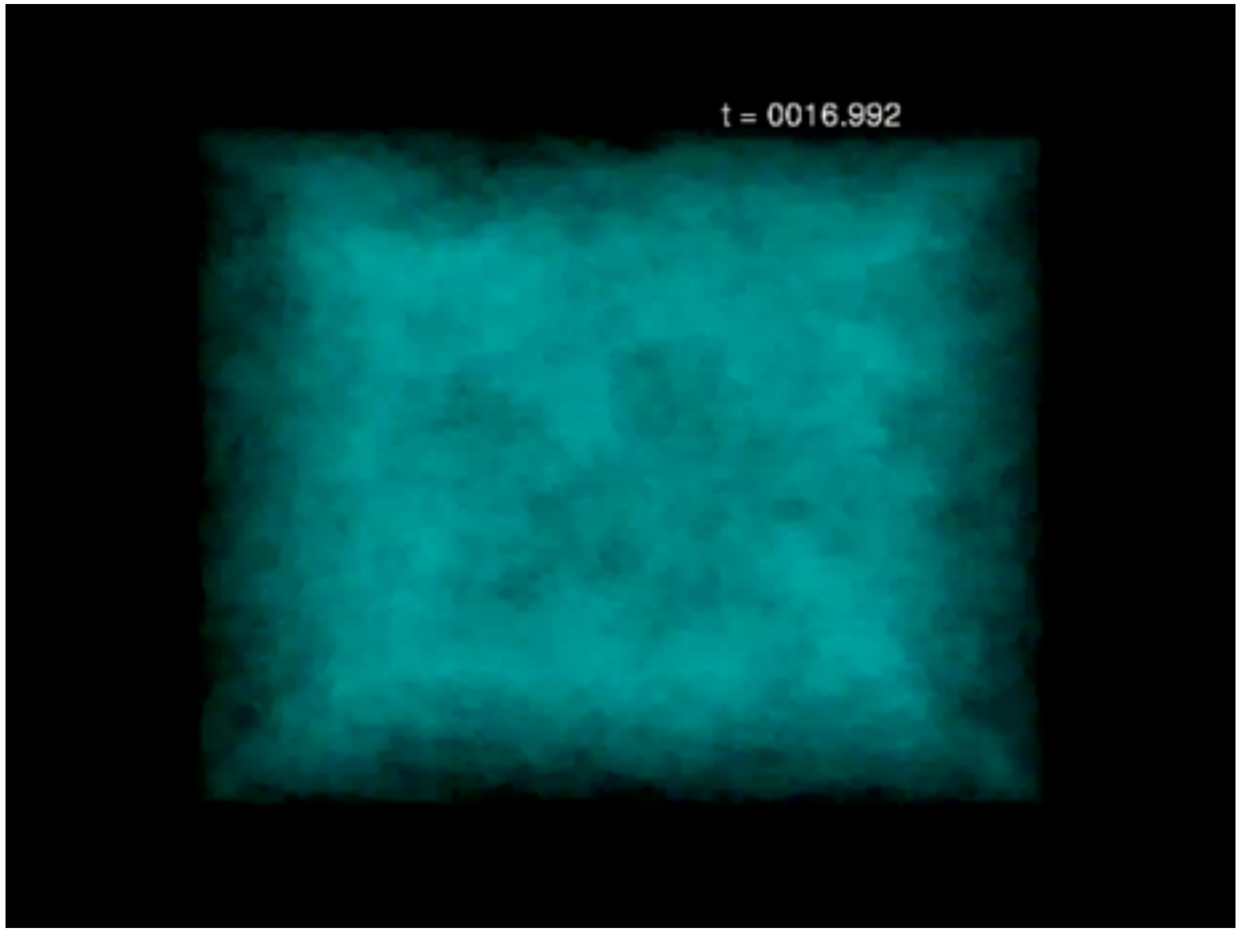
$$\mathcal{P}\left[\delta g_{\mu\nu}\left(x, t=0\right)\right]$$

1 parameter (for CMB)

$$\tau$$

Modelling the universe

Initially small fluctuations collapse to form galaxies, stars, etc.



That's it!

The rest is details.

Giving Thanks

- The non-linear Universe
  - GR is highly non-linear - inferring the state of the early universe would be like asking for the weather 100 million years ago based on the weather today.
  - No general classification of metrics - how to characterize initial conditions?
  - Shock waves, singularities, oh my!

Giving Thanks

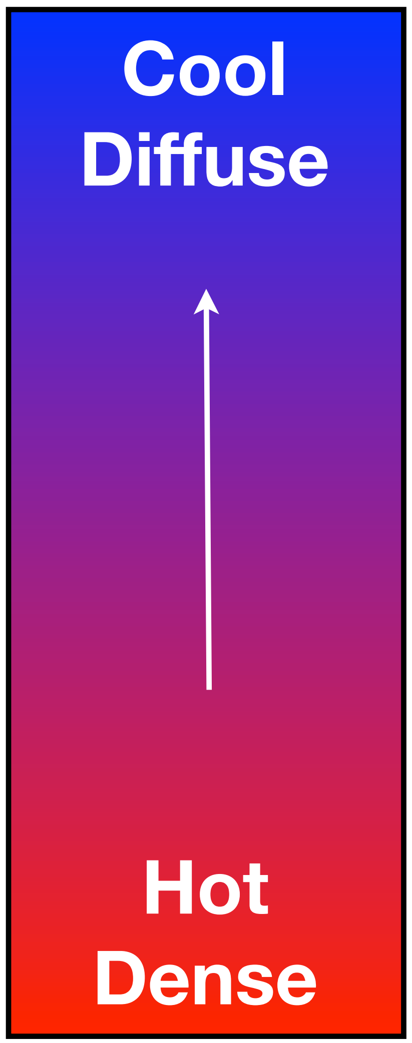
- The linear Universe
  - Simple evolution allows initial conditions to be inferred.
  - Background evolution and growth of structure can be analyzed separately.
  - Simple classification of initial conditions and metric degrees of freedom.
  - Physics on different scales evolves independently (Fourier modes independent).

The rest

Now for some details....

The homogeneous universe

The homogeneous universe



13.7 Billion Years: the present.

9.1 Billion Years: our sun ignites.

100 million years galaxies and first stars form.

380,000 years: neutral atoms form.

1 second: atomic nuclei form.

10^{-6} seconds: protons and neutrons form.

?Big Bang?

The homogeneous universe

- The metric in a flat, homogeneous, isotropic universe:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

comoving coordinates

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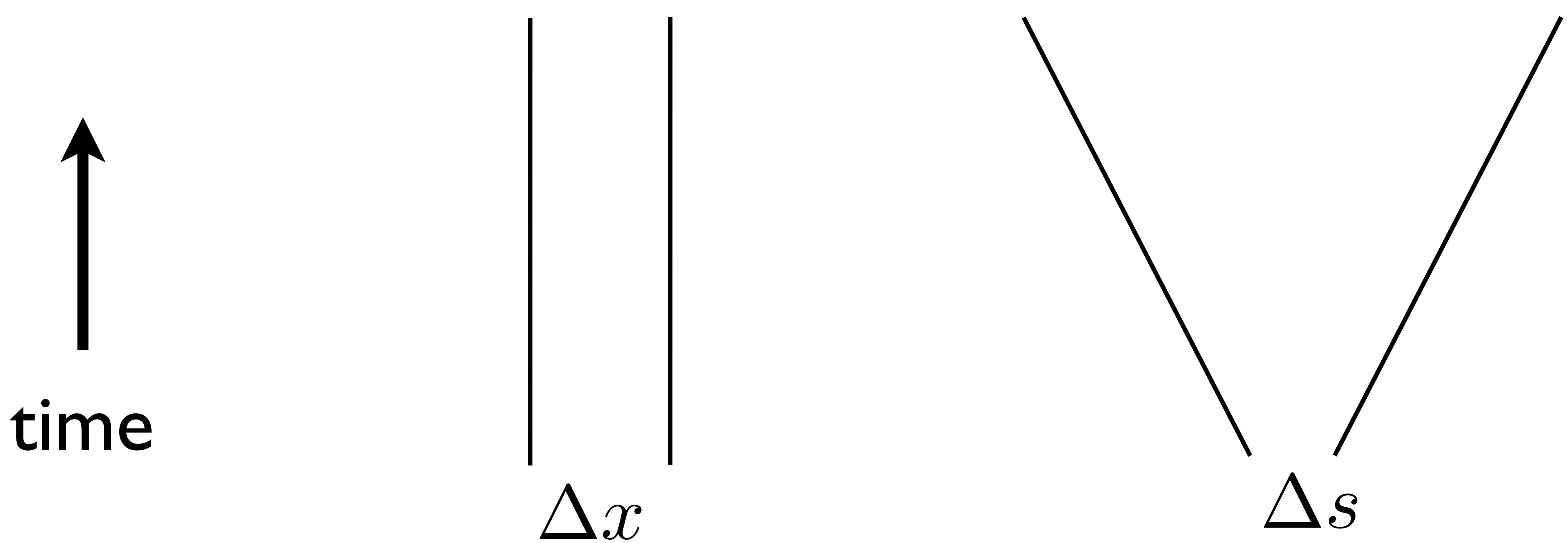

constant comoving distance = growing physical distance

The homogeneous universe

- Conformal time:

$$\eta = \int \frac{dt}{a(t)} \qquad ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right]$$

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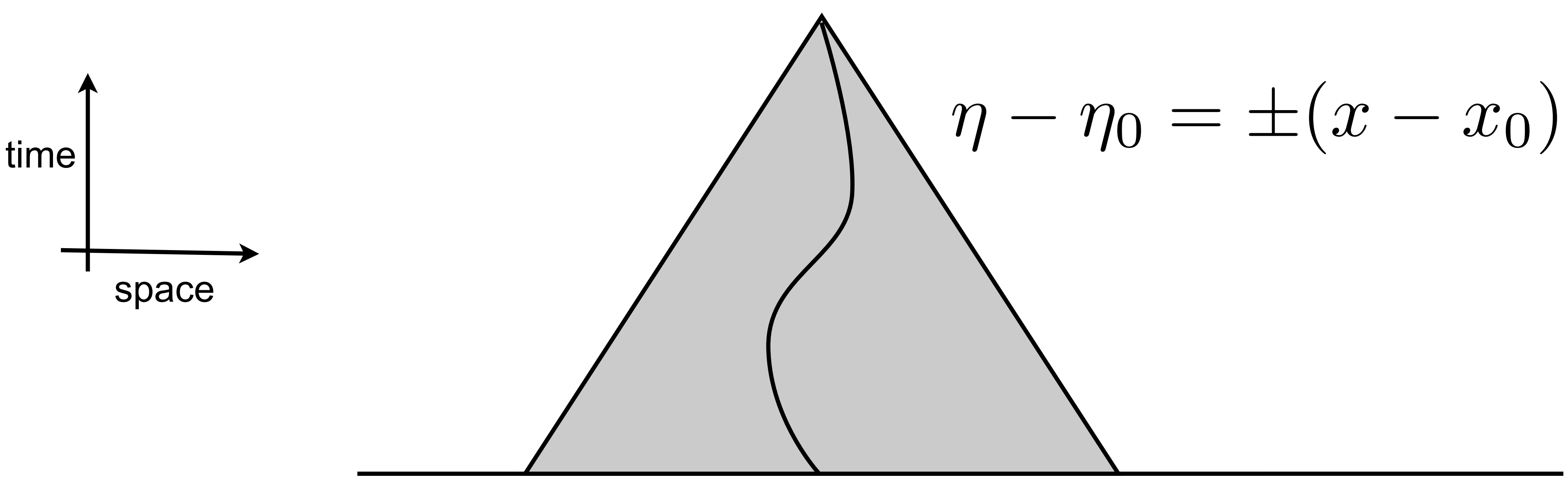

The big bang: when a(t)=0

The homogeneous universe

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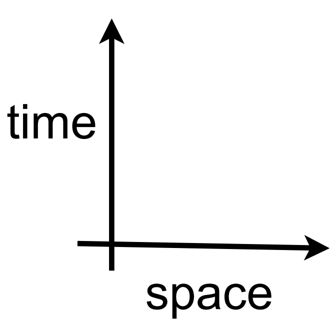
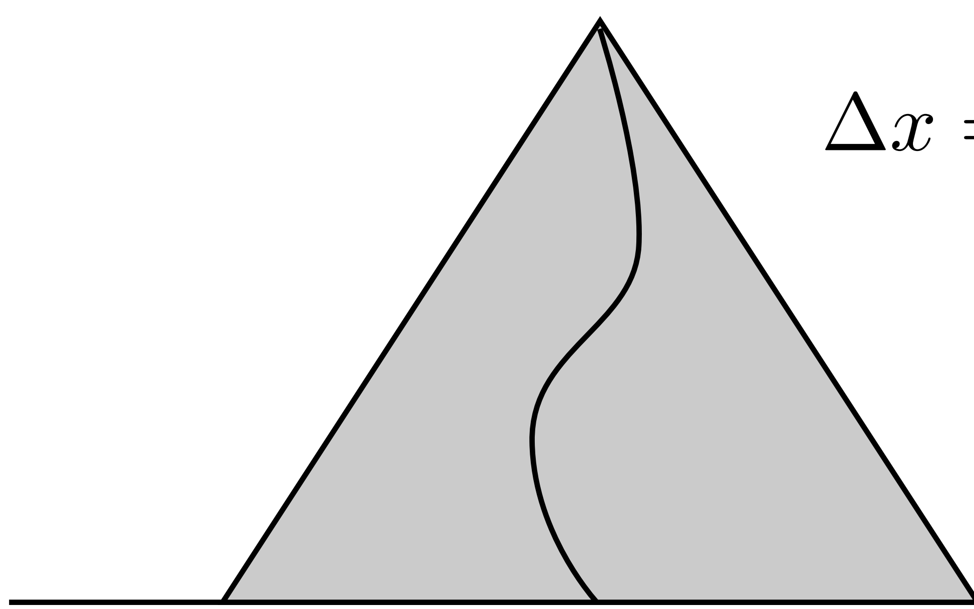
$$\eta = \int \frac{dt}{a(t)} \qquad\qquad ds^2 = a^2(\eta)\left[-d\eta^2 + \delta_{ij}dx^i dx^j\right]$$

particle horizon

$$\Delta x = \int \frac{dt}{a(t)} = \int_{a=0}^{a_0=1} \frac{d\ln(a)}{aH}$$

comoving horizon





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The homogeneous universe

- Equations of motion in a homogeneous universe:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \Longrightarrow \quad H^2 \equiv \left(\frac{\dot{a}}{a}\right) = \frac{8\pi G\rho}{3}$$

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$$p = w\rho$$

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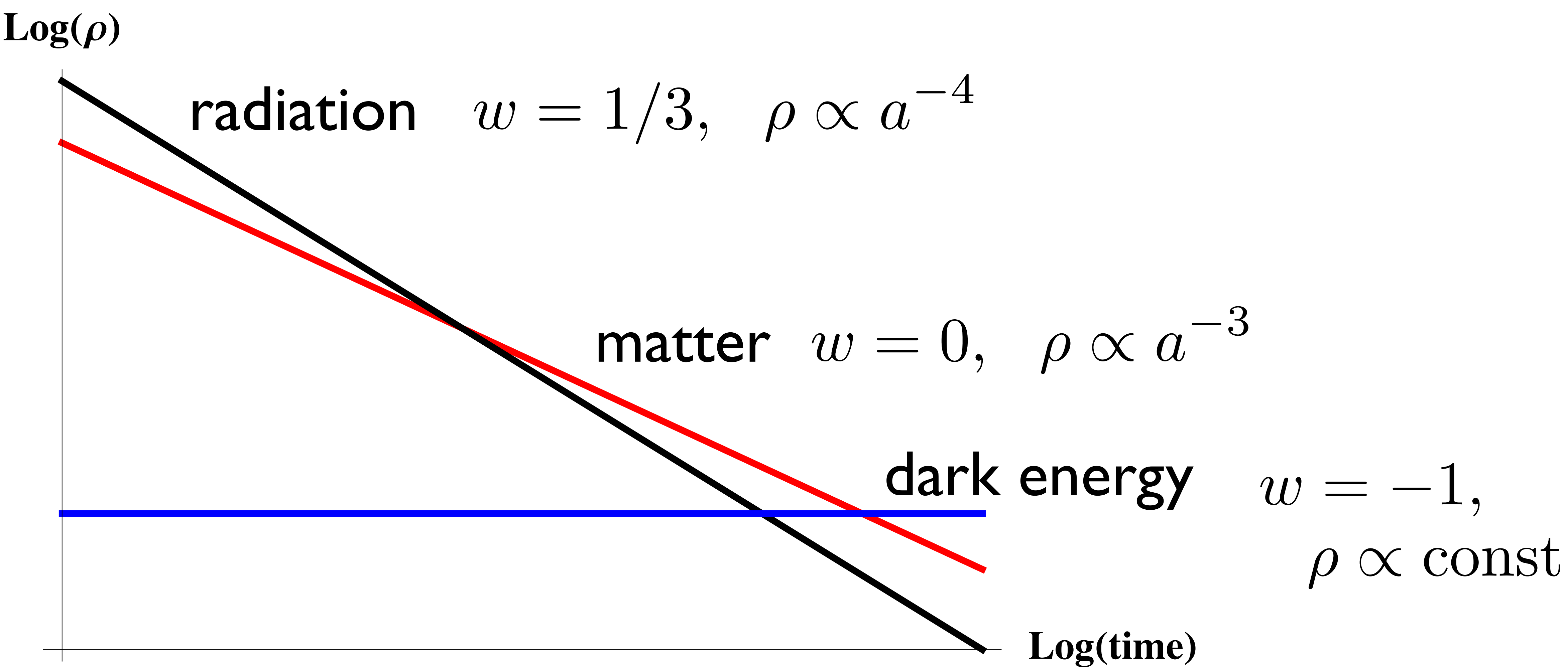
- Solutions:

$$\rho = \rho_0 a^{-3(1+w)} \qquad\qquad a(t) = a_0 t^{\frac{2}{3(1+w)}}$$

different fluids gravitate differently!

The homogeneous universe



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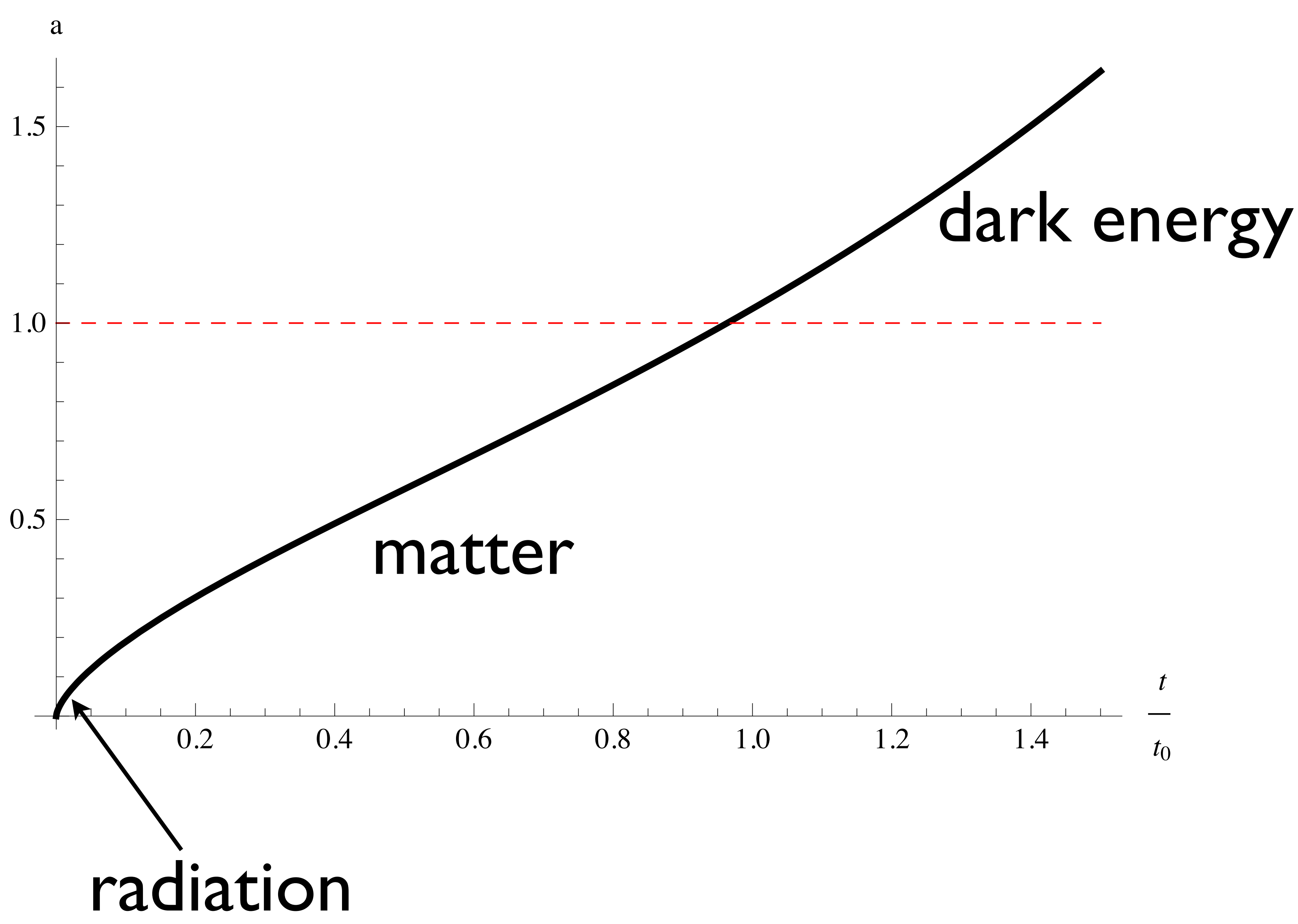
The homogeneous universe

- Evolution of the scale factor:



The homogeneous universe

- Energy budget:

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} \qquad \sum_i \Omega_i = 1$$

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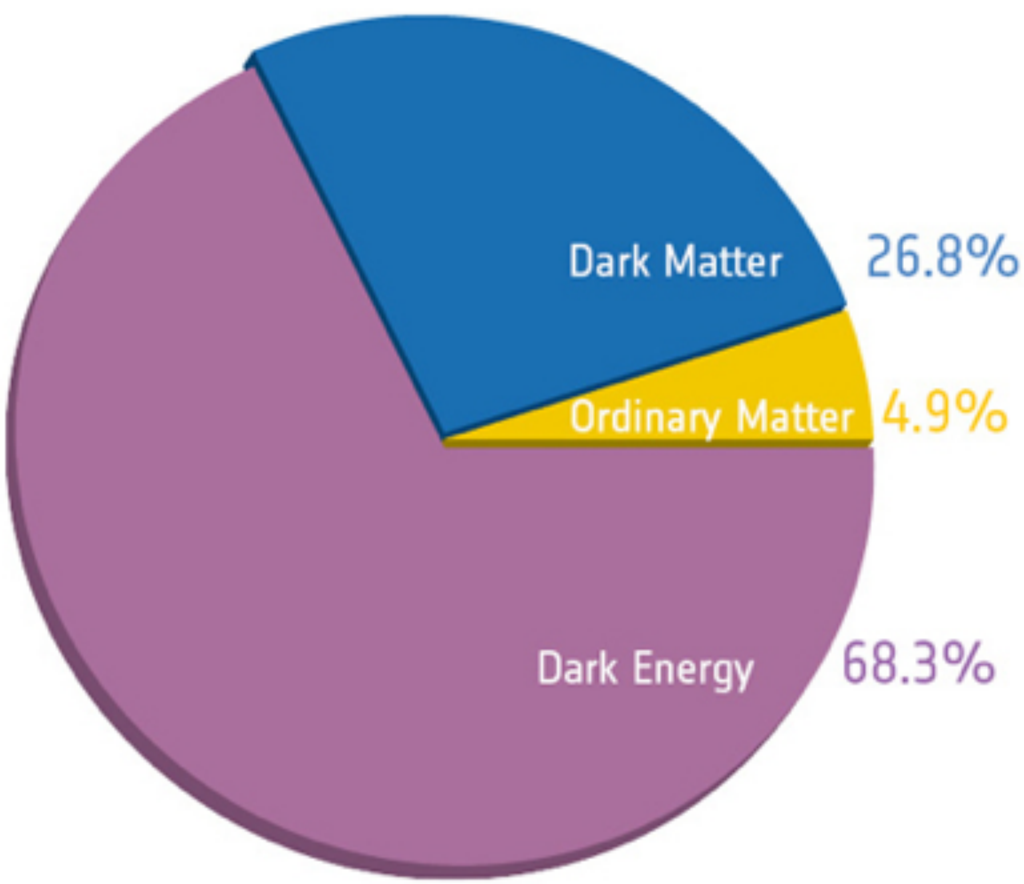


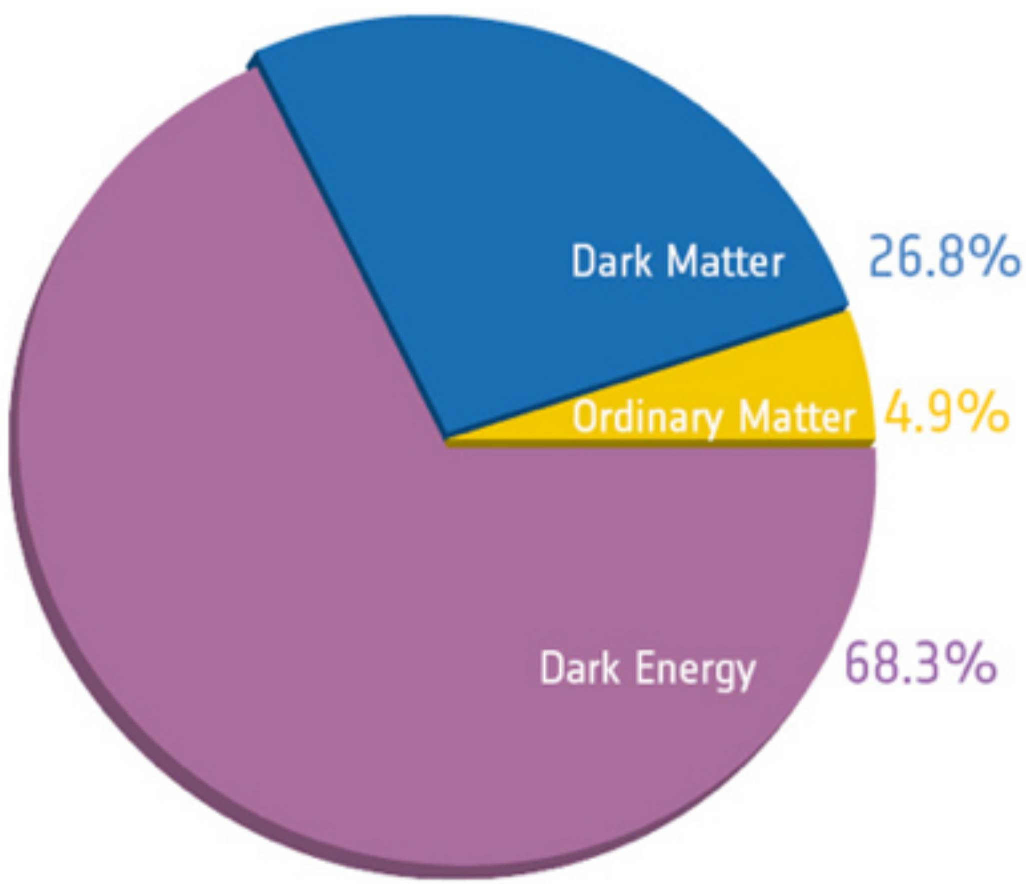
$$(\Omega_r \sim 10^{-4})$$

The homogeneous universe

- Energy budget:

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} \qquad \sum_i \Omega_i = 1$$

$$H_0 = 100h \frac{\text{km}}{\text{s Mpc}}$$
$$= \frac{h}{3000 \text{ Mpc}}$$



$$(\Omega_r \sim 10^{-4})$$

$$1 \text{ Mpc} = 3 \times 10^{23} \text{ meters}$$

$$1 \text{ Mpc} = 3.3 \times 10^6 \text{ Lyr}$$

The homogeneous universe

- Redshift: $z = \dfrac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm obs}}$

$$a = \frac{1}{1+z}$$

$$a = 1, \quad z = 0$$

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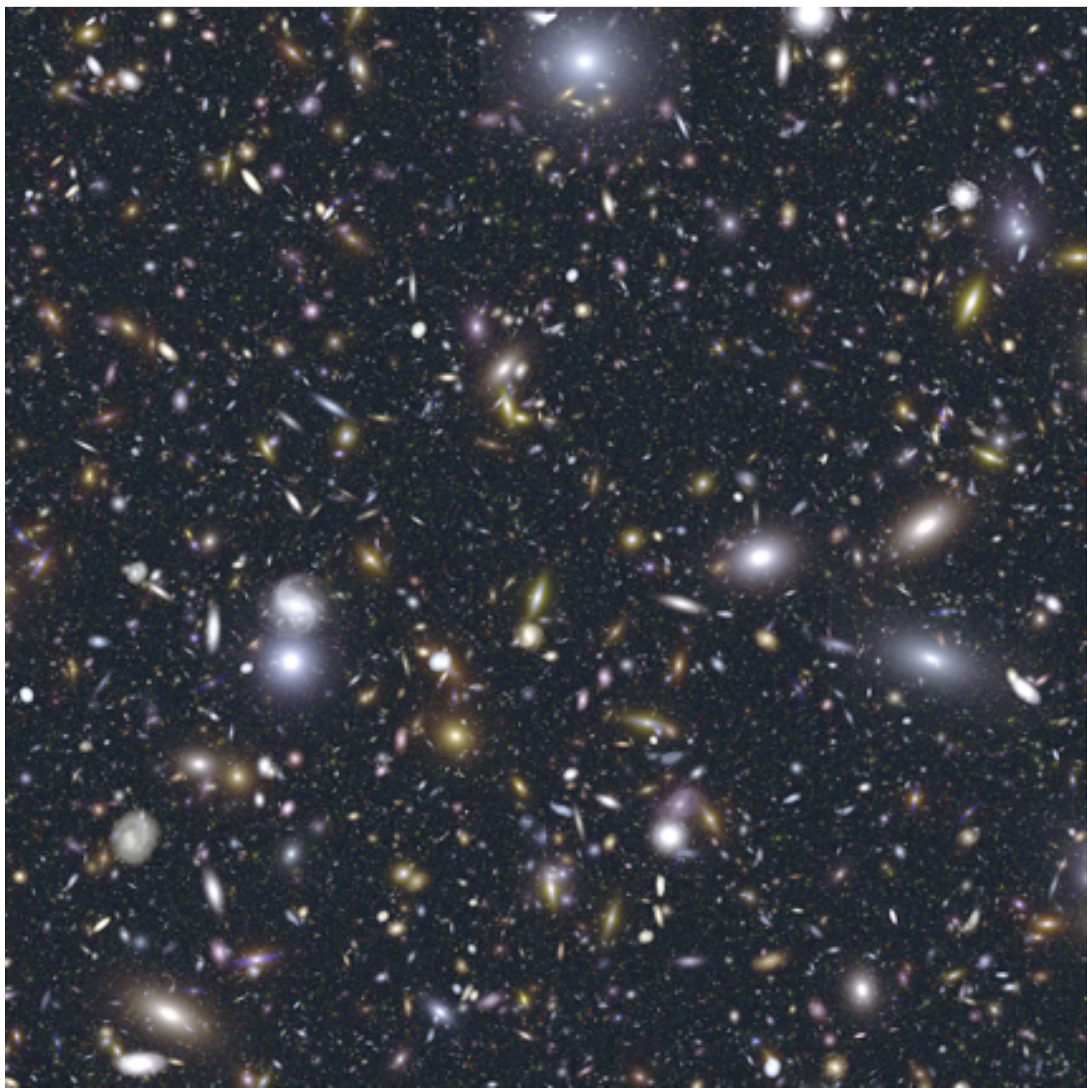
$$\begin{aligned}
z_{\rm eq} &= 3400, \\
z_* &= 1090, \\
z_{\rm re} &\sim 11, \\
z_{\rm gal} &\sim 11 - 12, \\
z_{\rm surveys} &\lesssim 1, \\
z_\Lambda &= .28, \\
z_{\rm Virgo} &= .003
\end{aligned}$$

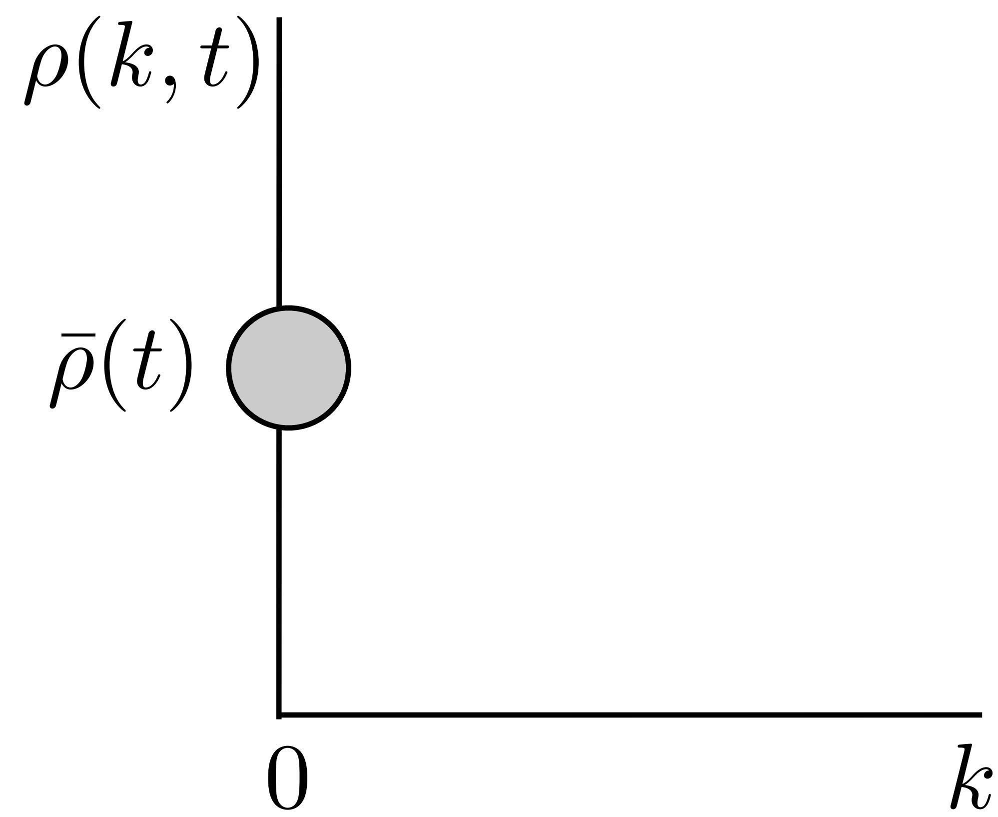
The Inhomogeneous Universe

- There is structure in the Universe:



- In Fourier space:

$$\rho(x,t) = \frac{1}{V}\sum \rho(k,t)e^{i\vec{k}\cdot\vec{x}}$$



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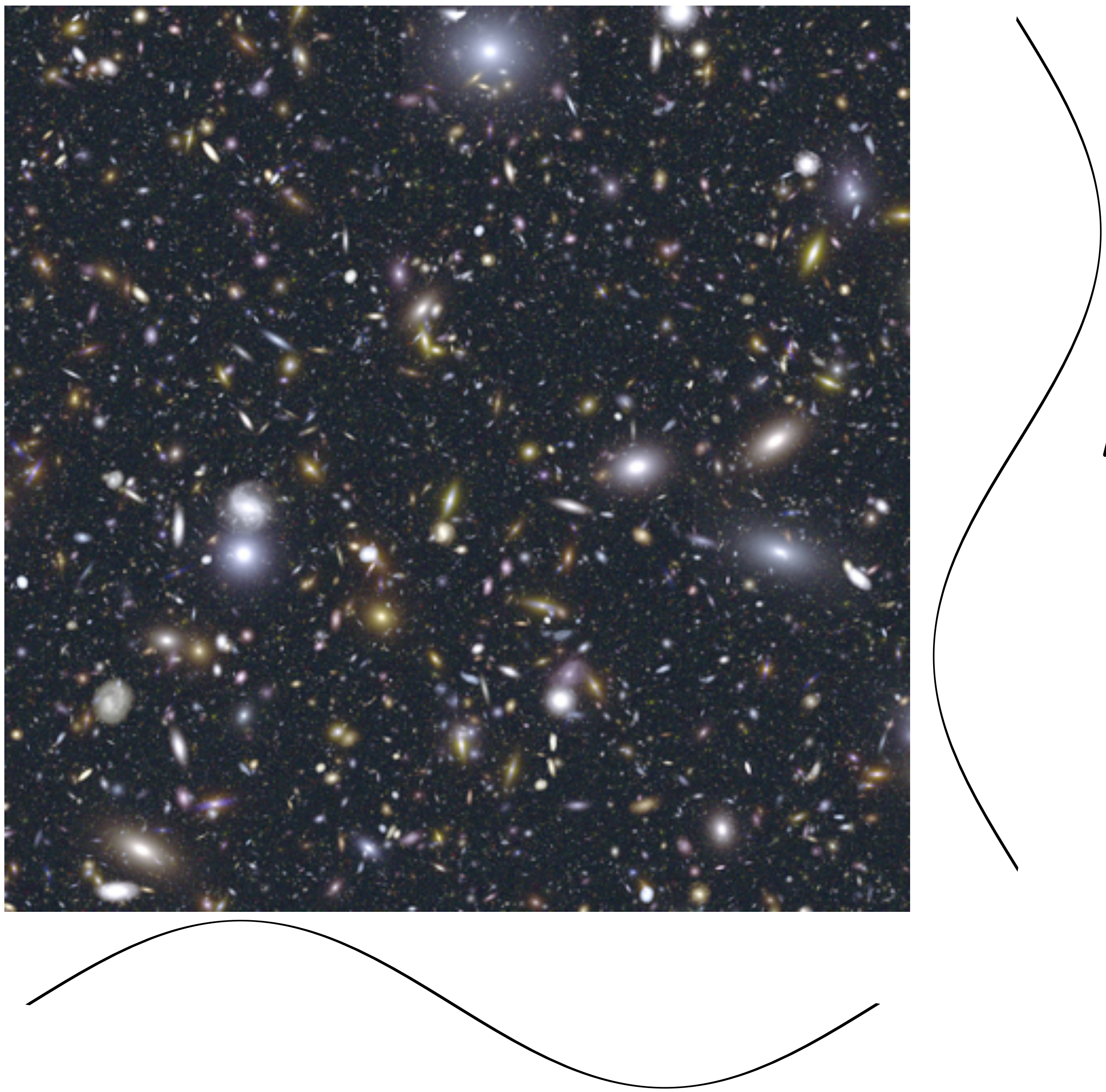

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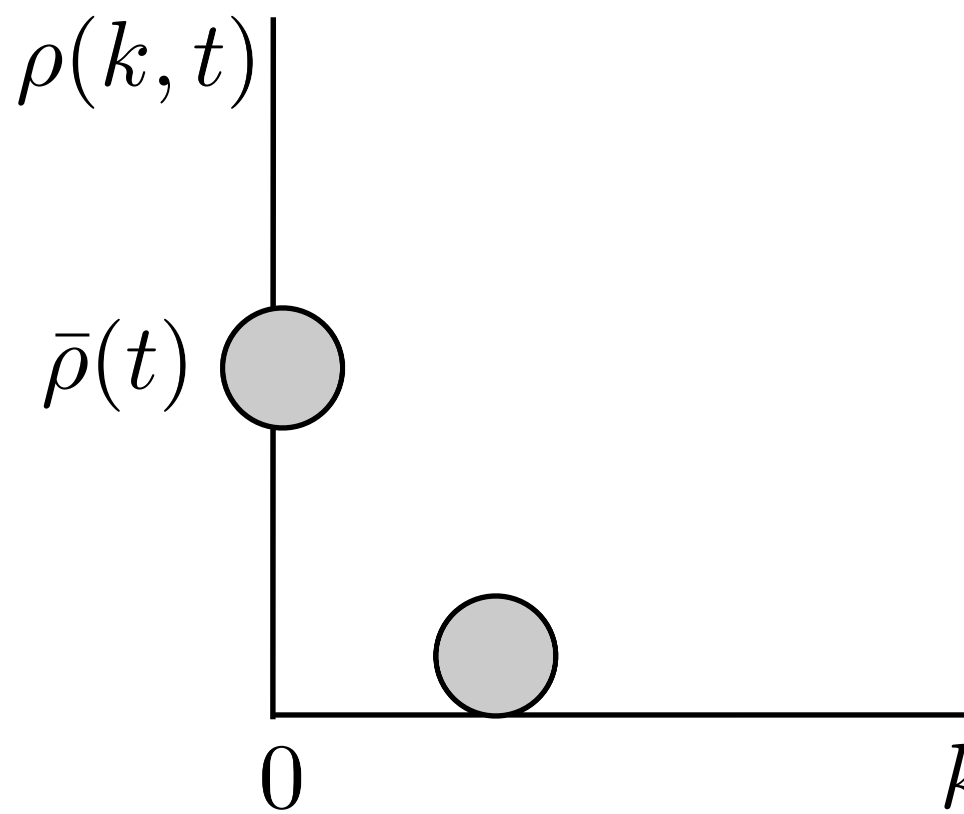

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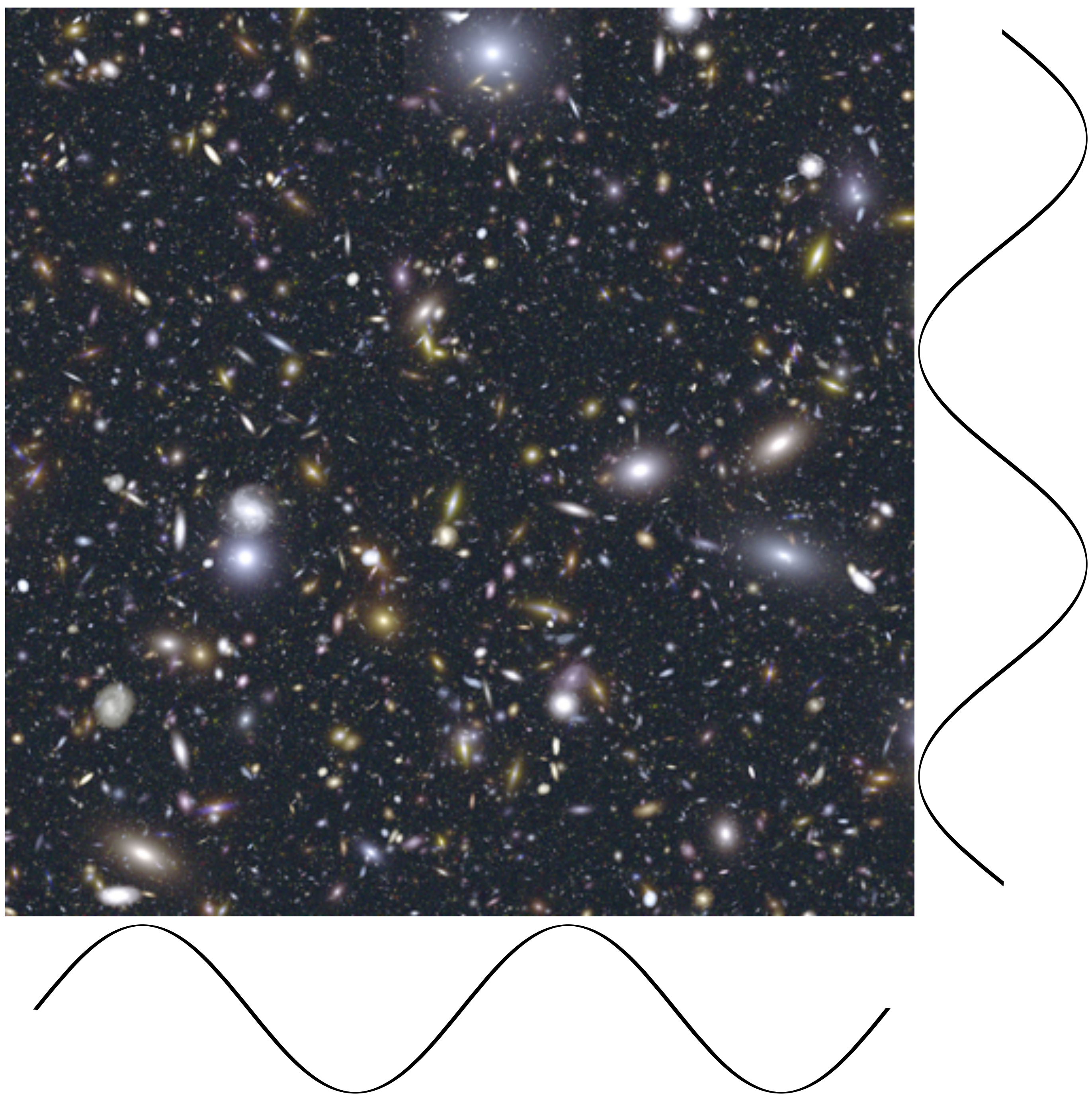

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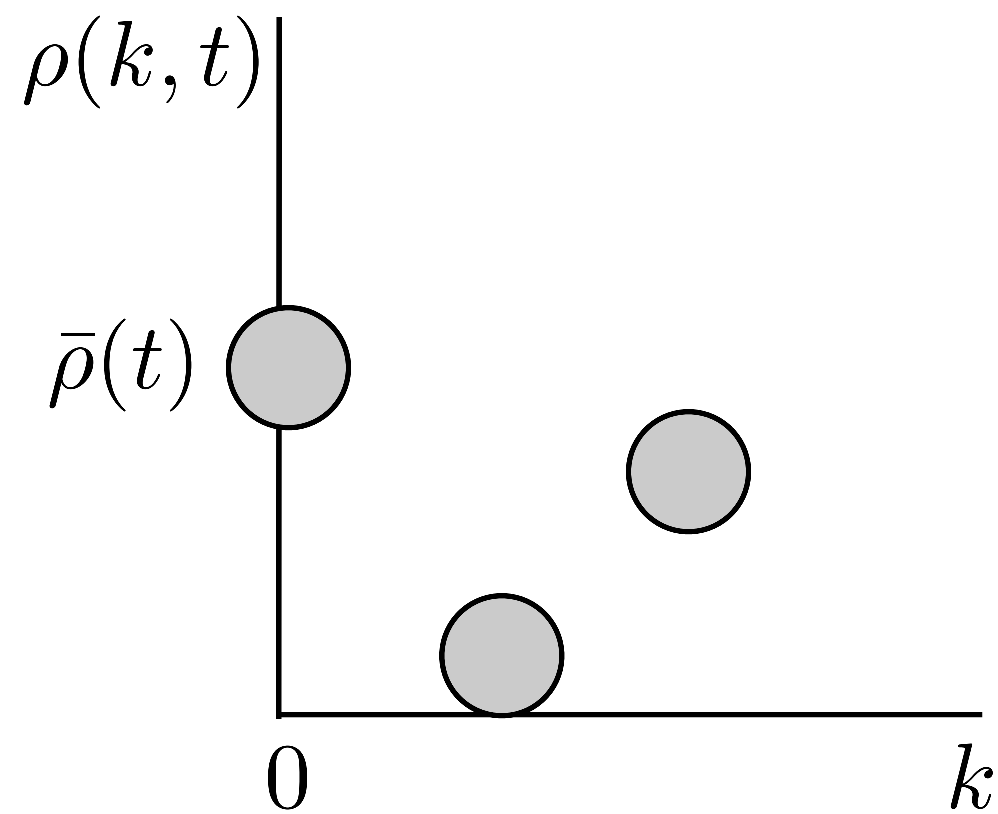

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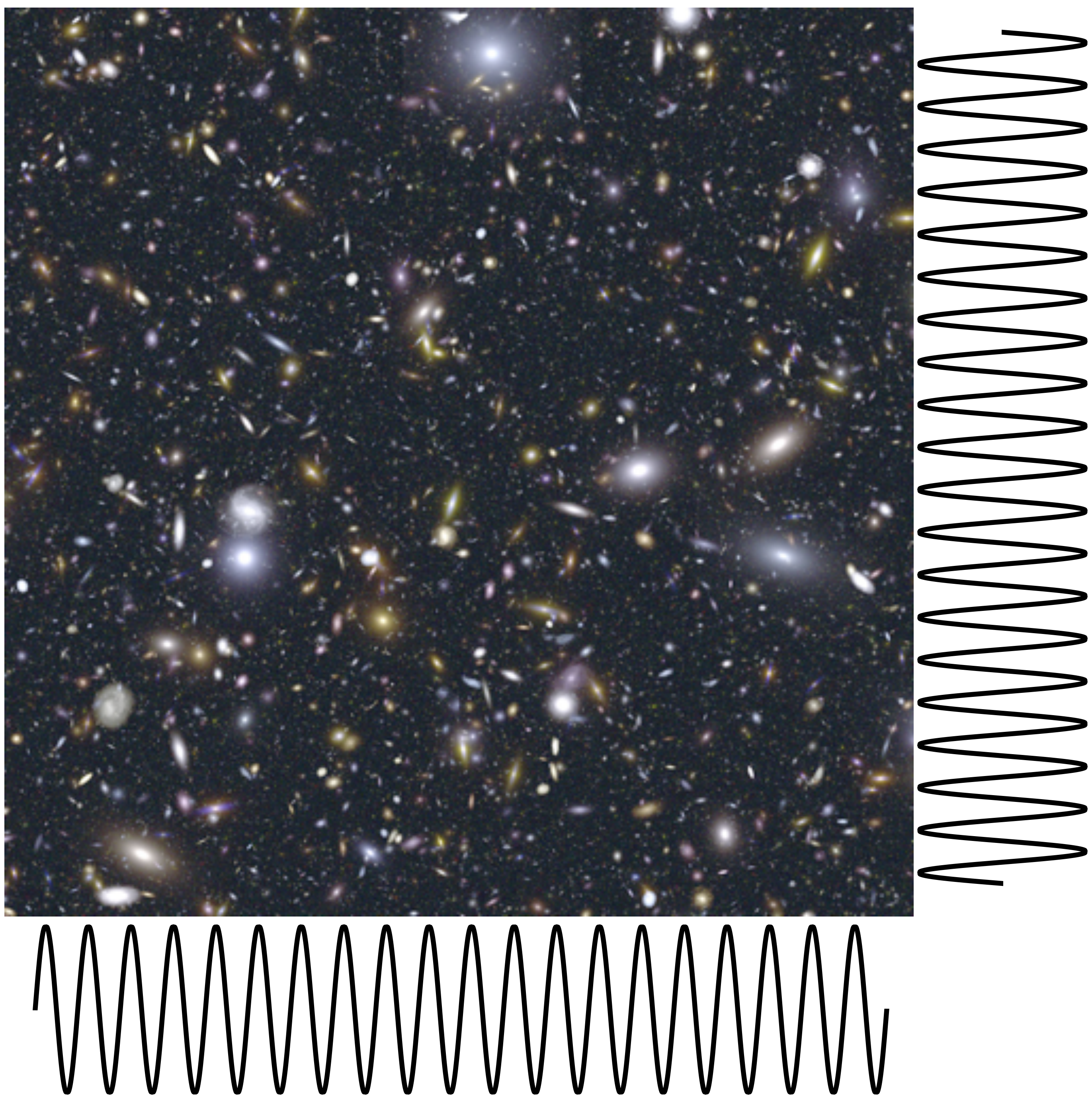

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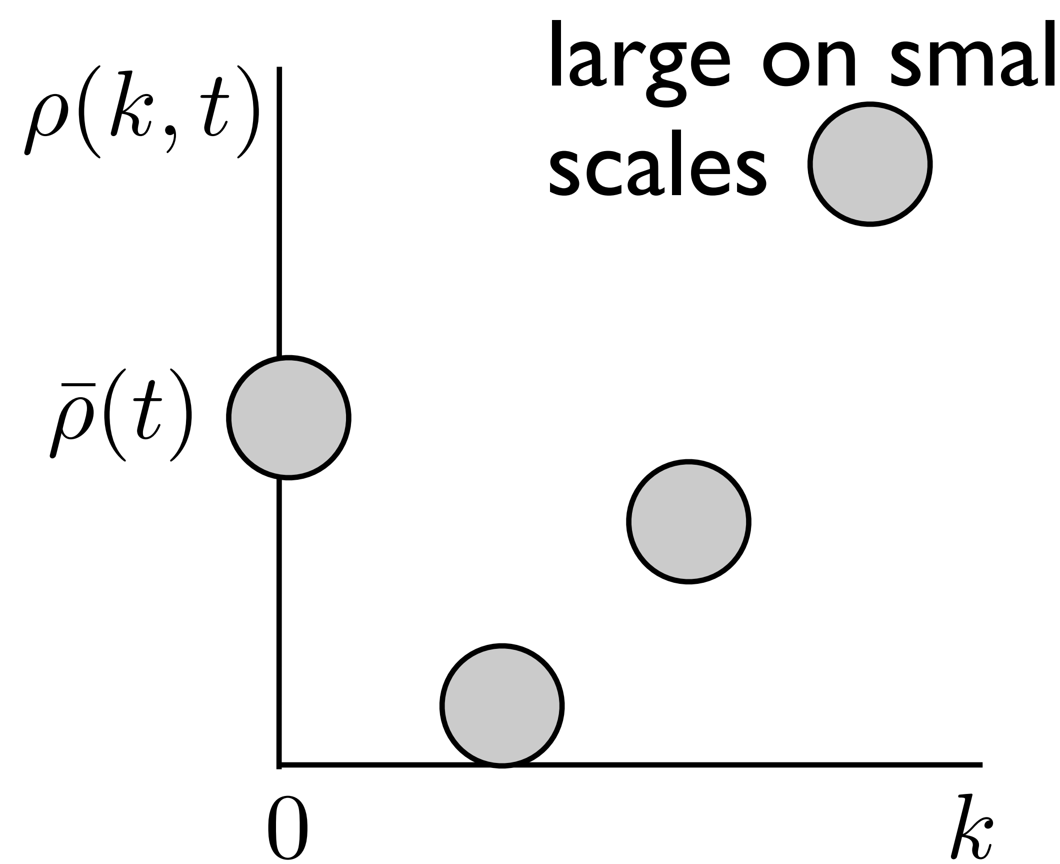

The Inhomogeneous Universe

- There is structure in the Universe:

Structure: $\frac{\delta\rho}{\bar{\rho}} \gtrsim 1$

$\mathcal{O}(10^6)$ galaxies

$\mathcal{O}(10^3)$ clusters

$\mathcal{O}(1)$ superclusters

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- There is structure on all scales which have had a chance to undergo gravitational collapse.

- The largest structures in the Universe define a scale above which the fluctuations in density are linear:

$$k = .1 \ - .01\mathrm{Mpc}^{-1} \qquad \lambda \sim 1\% \text{ of observable Universe}$$

wave number

The Inhomogeneous Universe

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Luminous:
Baryons
Photons

Semi-Luminous:
Neutrinos

The Inhomogeneous Universe

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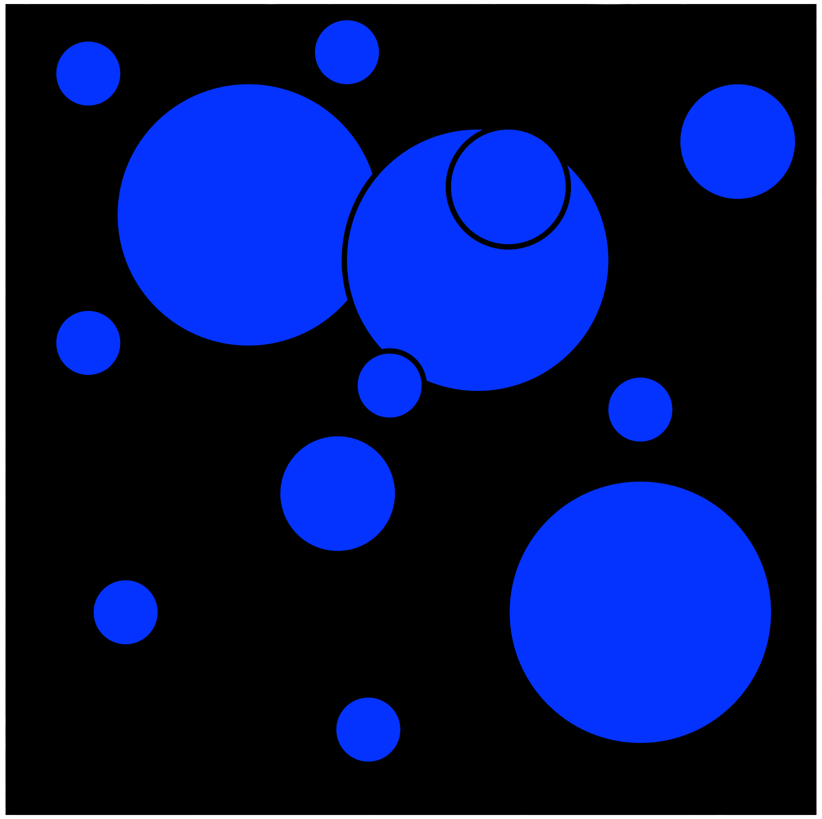

Luminous:
Baryons
Photons

Semi-Luminous:
Neutrinos

Dark:
Dark Matter
Dark Energy

Gravitational Instability

- For non-relativistic matter in flat space:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

continuity

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} + \frac{\nabla p}{\rho} + \nabla \phi = 0$$

Euler

$$\nabla^2 \phi = 4\pi G \rho$$

Poisson

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continuity

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Poisson

- Linearize:

$$\rho = \bar{\rho} + \delta\rho \qquad \phi = \bar{\phi} + \delta\phi \qquad \vec{v} = \bar{\vec{v}} + \delta\vec{v}$$

$$p = \bar{p} + c_s^2 \delta\rho$$

$$c_s^2 = \frac{\partial p}{\partial \rho} = w$$

Gravitational Instability

- Linearized equation of motion:

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_s^2 \nabla^2 \delta\rho - 4\pi G \bar{\rho} \delta\rho = 0$$

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- Fourier transform:

$$\delta\rho(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \delta\rho(t, k) e^{i\vec{k}\cdot\vec{x}}$$

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$$\delta\rho(t, k) = A \exp\left(i\omega(k)t\right) + B \exp\left(-i\omega(k)t\right)$$

$$\omega(k) = \sqrt{k^2 c_s^2 - 4\pi G \bar{\rho}}$$

Gravitational Instability

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- Jeans scale: competition between pressure and gravity

$$\lambda_J = \frac{2\pi}{k_J} = c_s \left(\frac{\pi}{G\bar{\rho}}\right)^{1/2}$$

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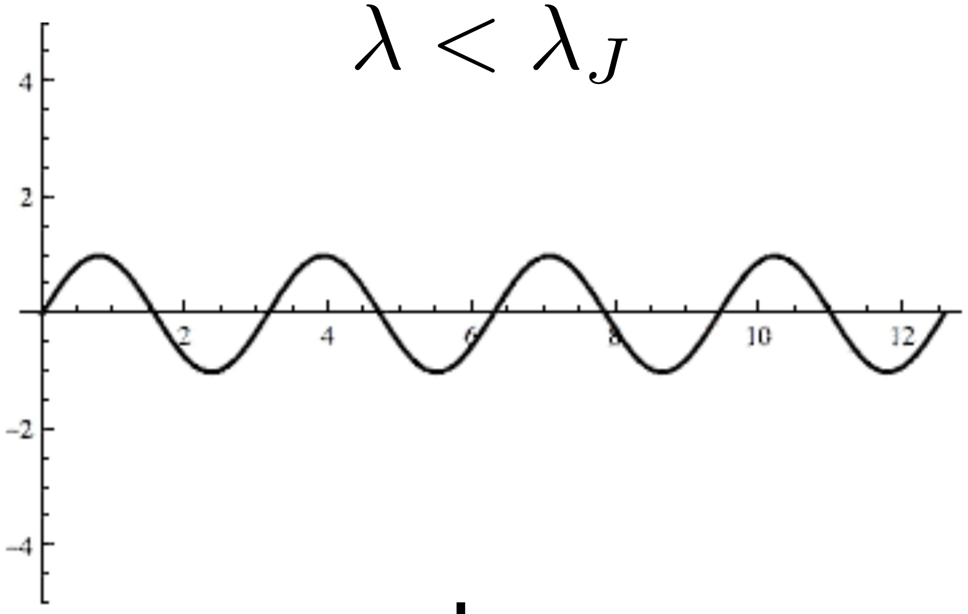
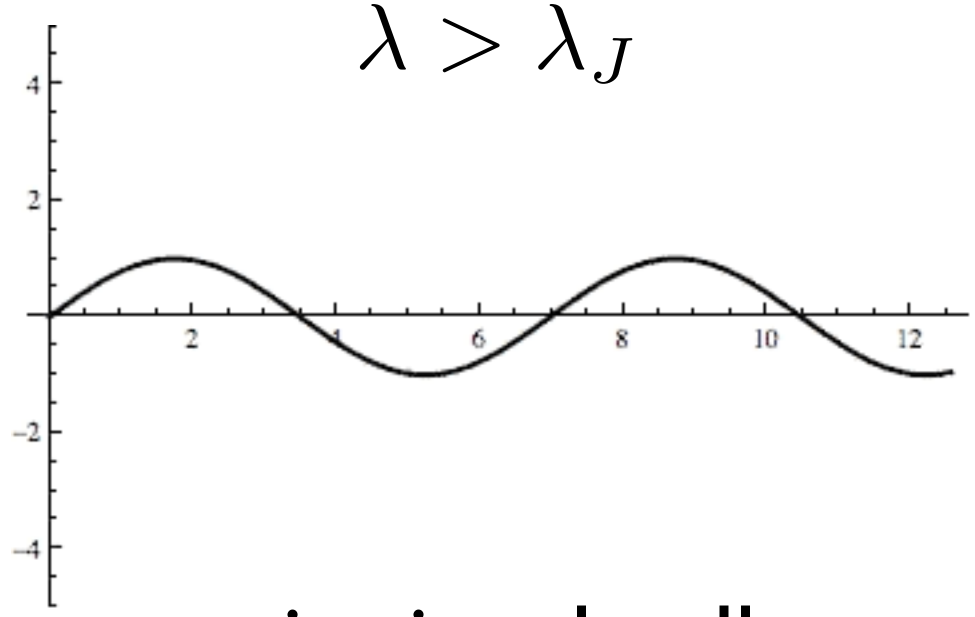
$\lambda < \lambda_J$



sound waves

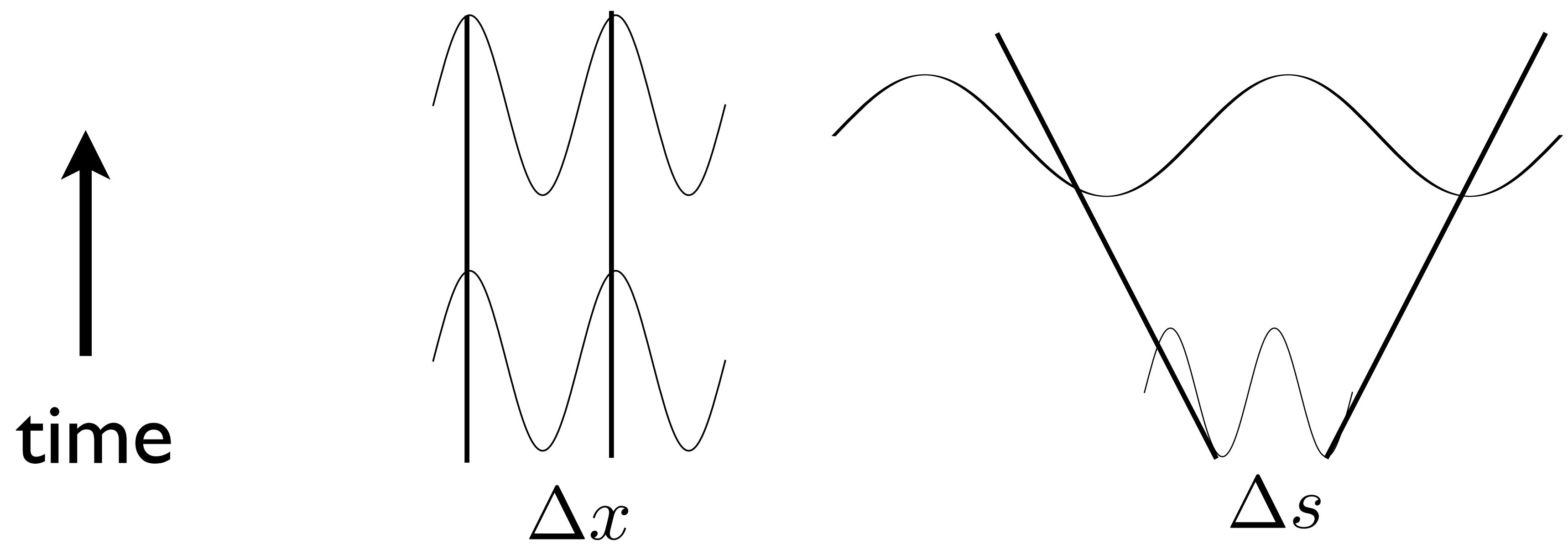
$\lambda > \lambda_J$



gravitational collapse

Gravitational Instability

- In an expanding universe waves are stretched:



$$\lambda_{\rm ph} = a(t)\lambda_{\rm com}$$

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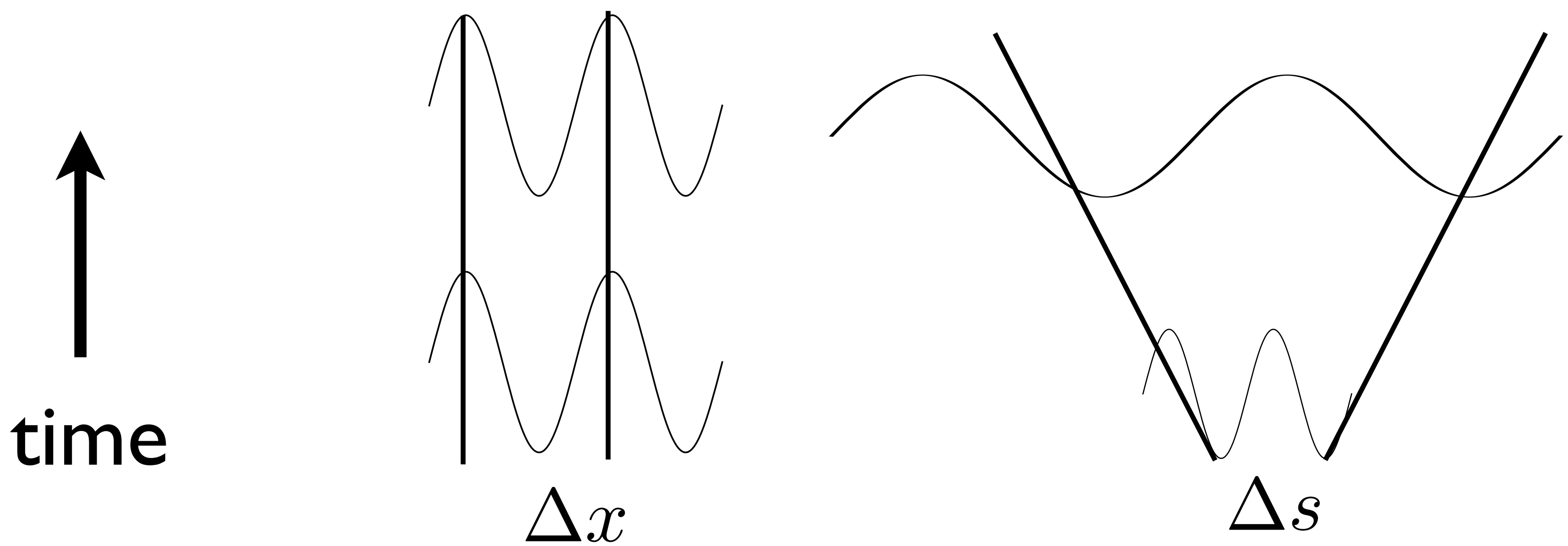

$$\lambda_{\rm ph} = a(t)\lambda_{\rm com}$$

- Only gravitationally bound (non-linear) structures separate from the Hubble flow.

- Expansion inhibits collapse.

Gravitational Instability

$t_{\text{coll}} \sim (4\pi G\bar{\rho})^{-1/2} \sim H^{-1}$ Expansion will be relevant!

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$c_s^2 = 0$

radiation	matter	dark energy
$\frac{\delta\rho}{\bar{\rho}} \propto \log(a)$	$\frac{\delta\rho}{\bar{\rho}} \propto a$	$\frac{\delta\rho}{\bar{\rho}} \propto \mathrm{const.}$

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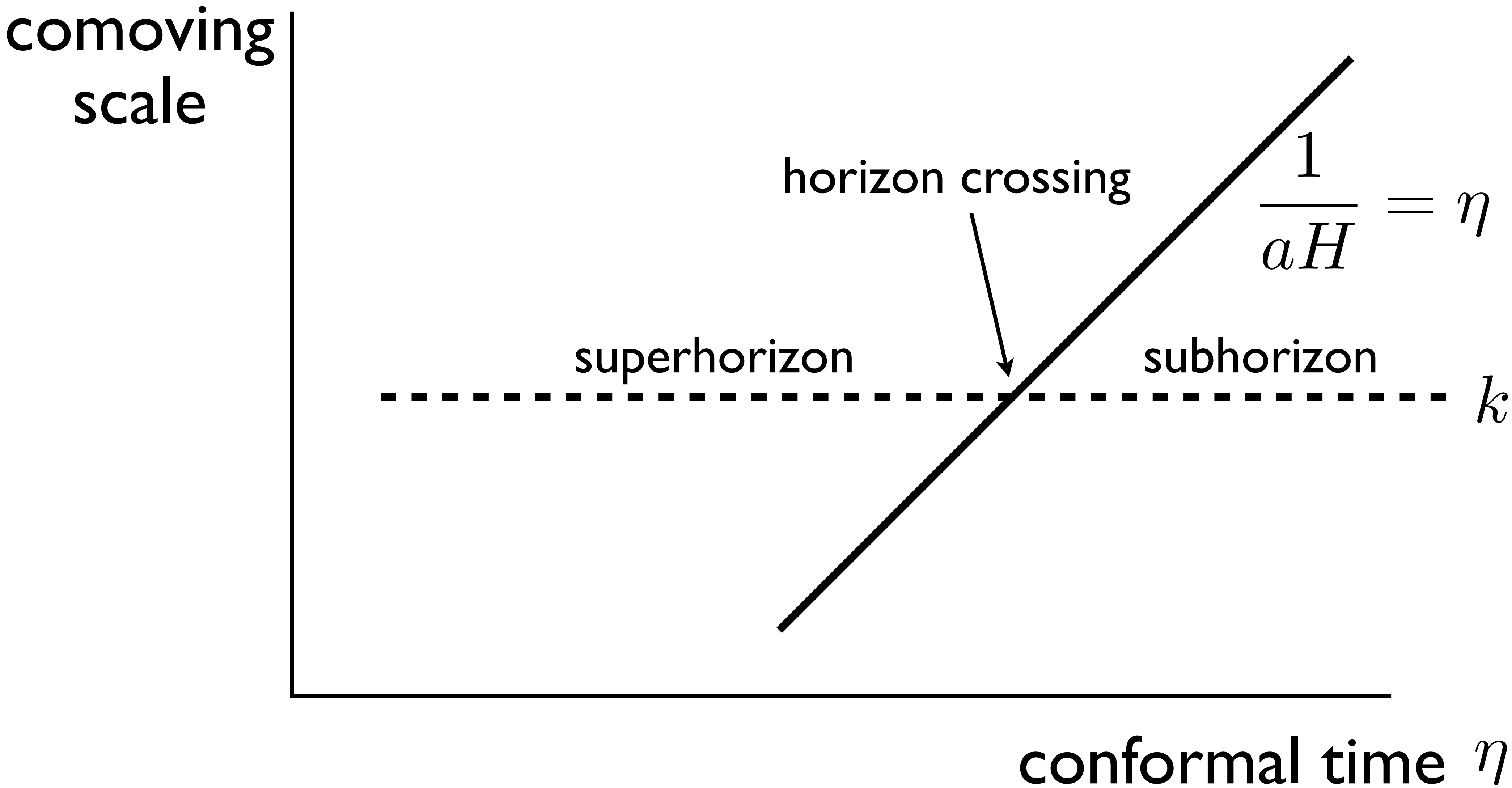
$c_s^2 = 0$

radiation	matter	dark energy
$\frac{\delta\rho}{\bar{\rho}} \propto \log(a)$	$\frac{\delta\rho}{\bar{\rho}} \propto a$	$\frac{\delta\rho}{\bar{\rho}} \propto \text{const.}$

To have structure, need: $\frac{\delta\rho}{\bar{\rho}}(t_{\rm eq}) \gtrsim 3\times 10^{-4}$

Gravitational Instability

- An important scale: comoving horizon $\dfrac{1}{aH}$
- Horizon crossing: $k = aH$



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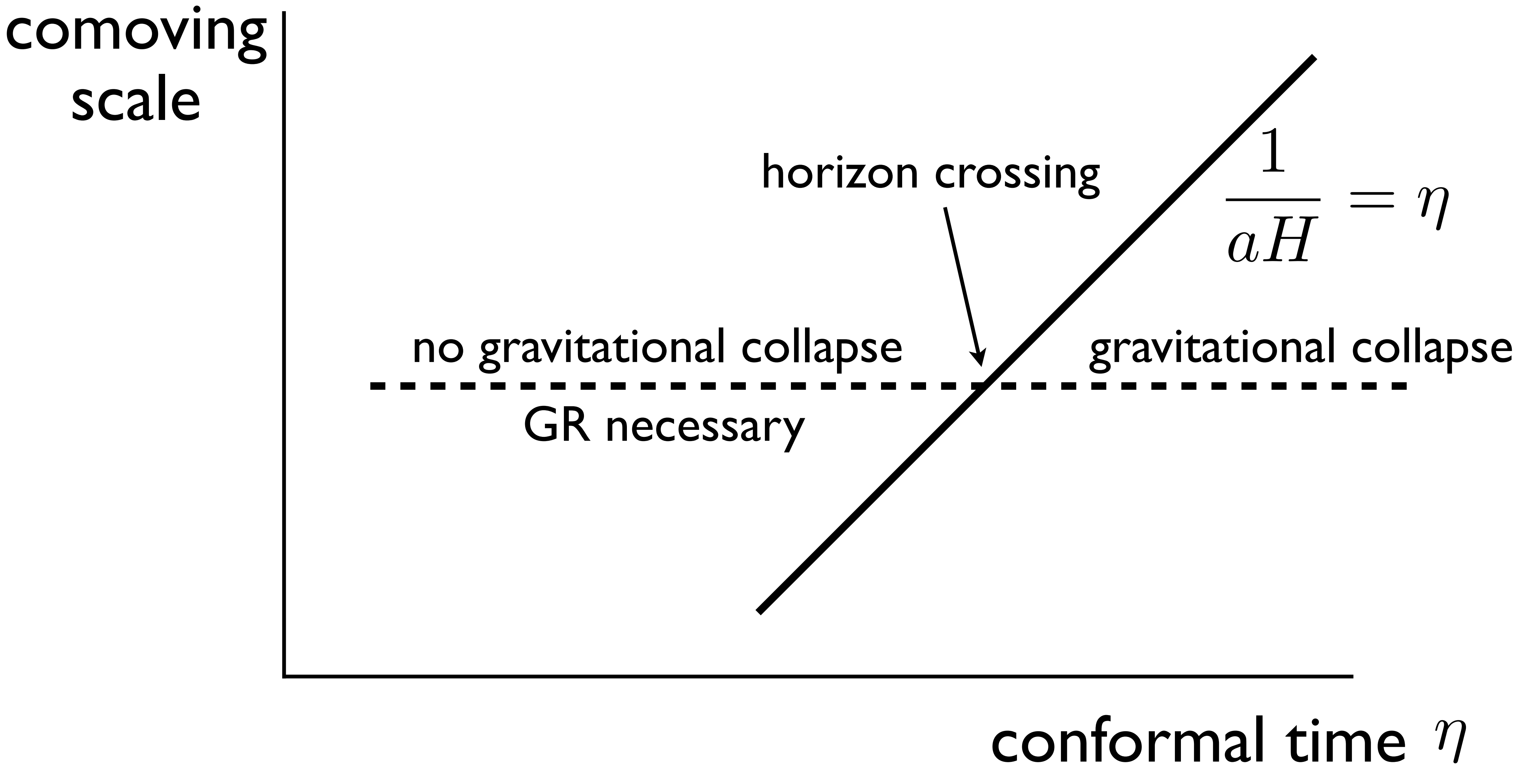

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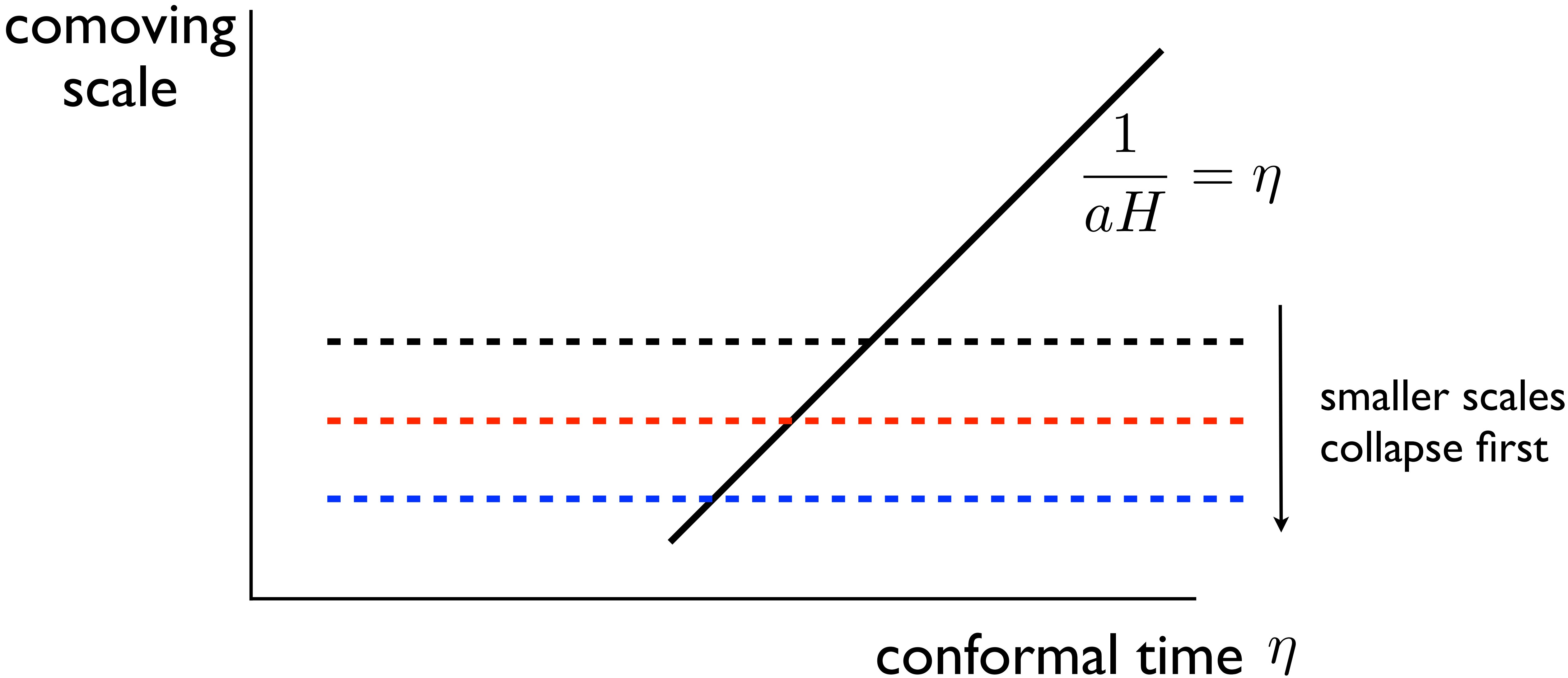

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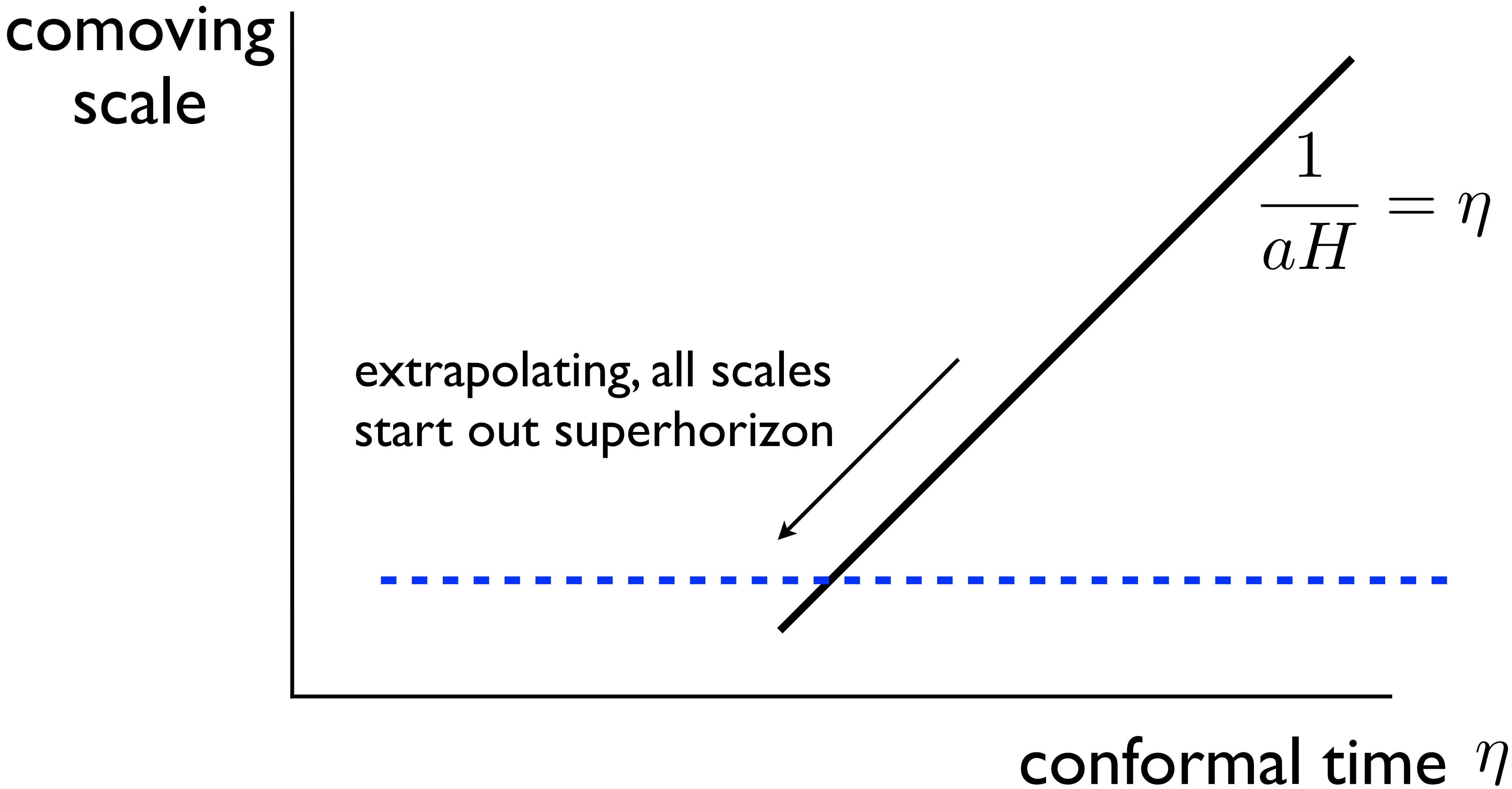

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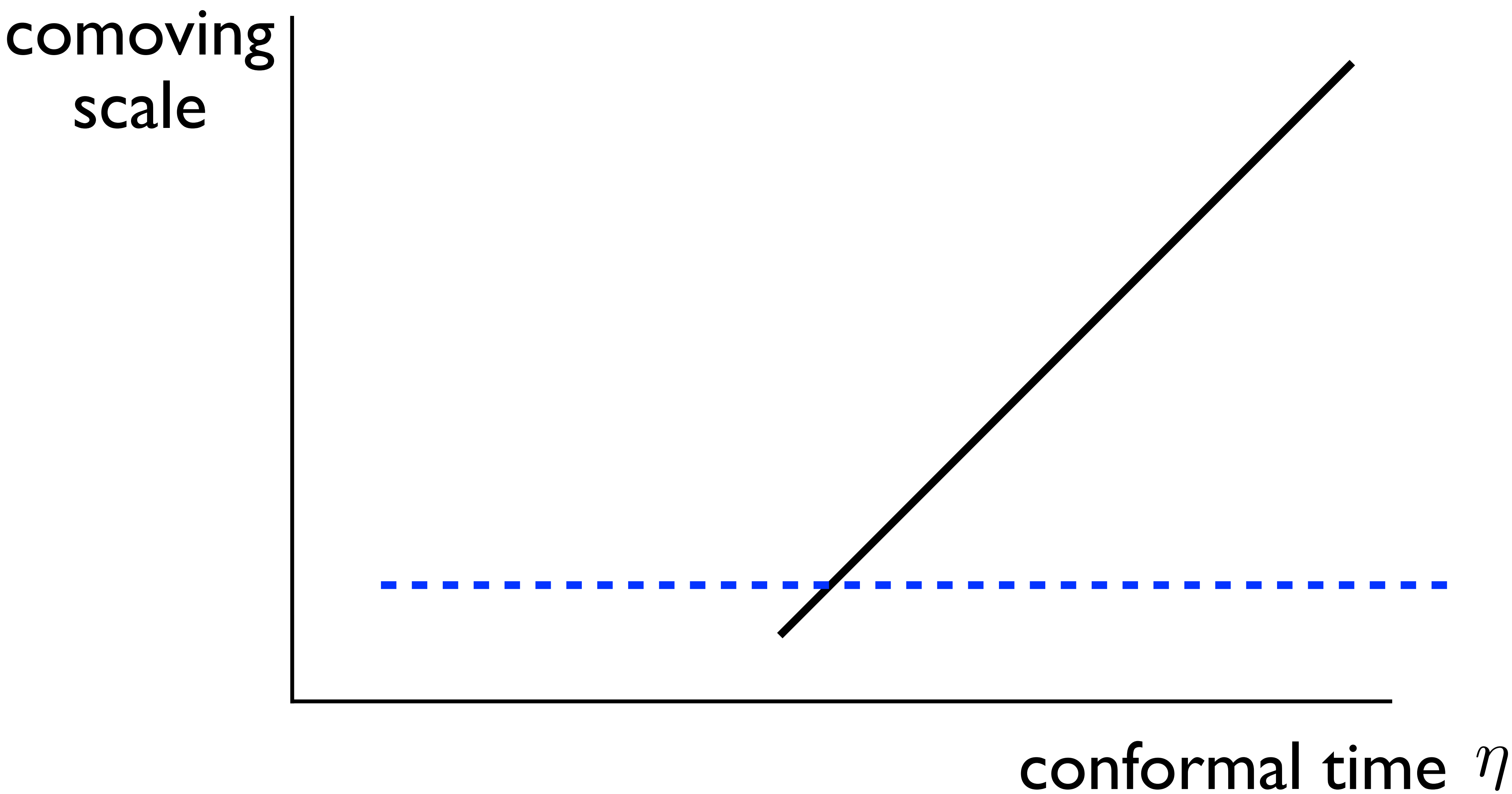

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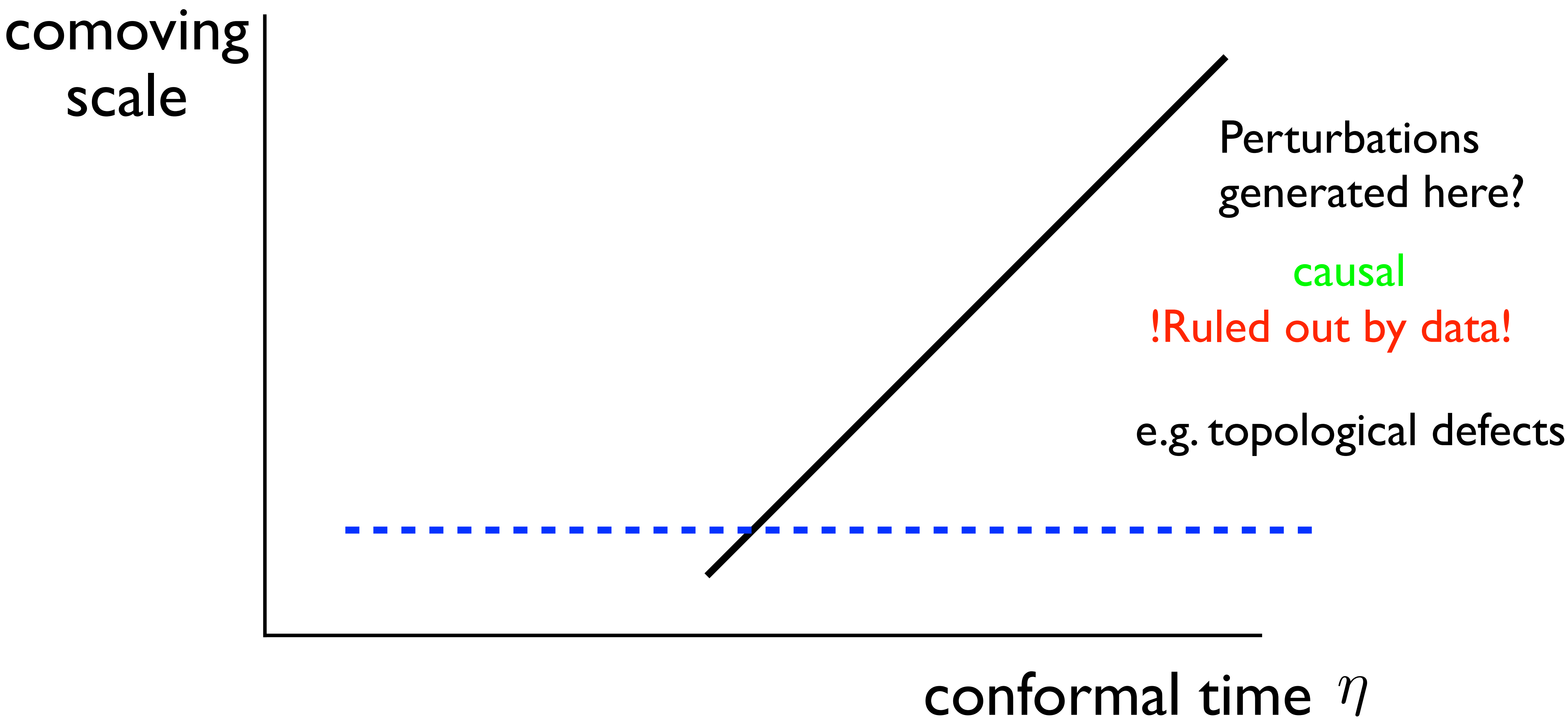

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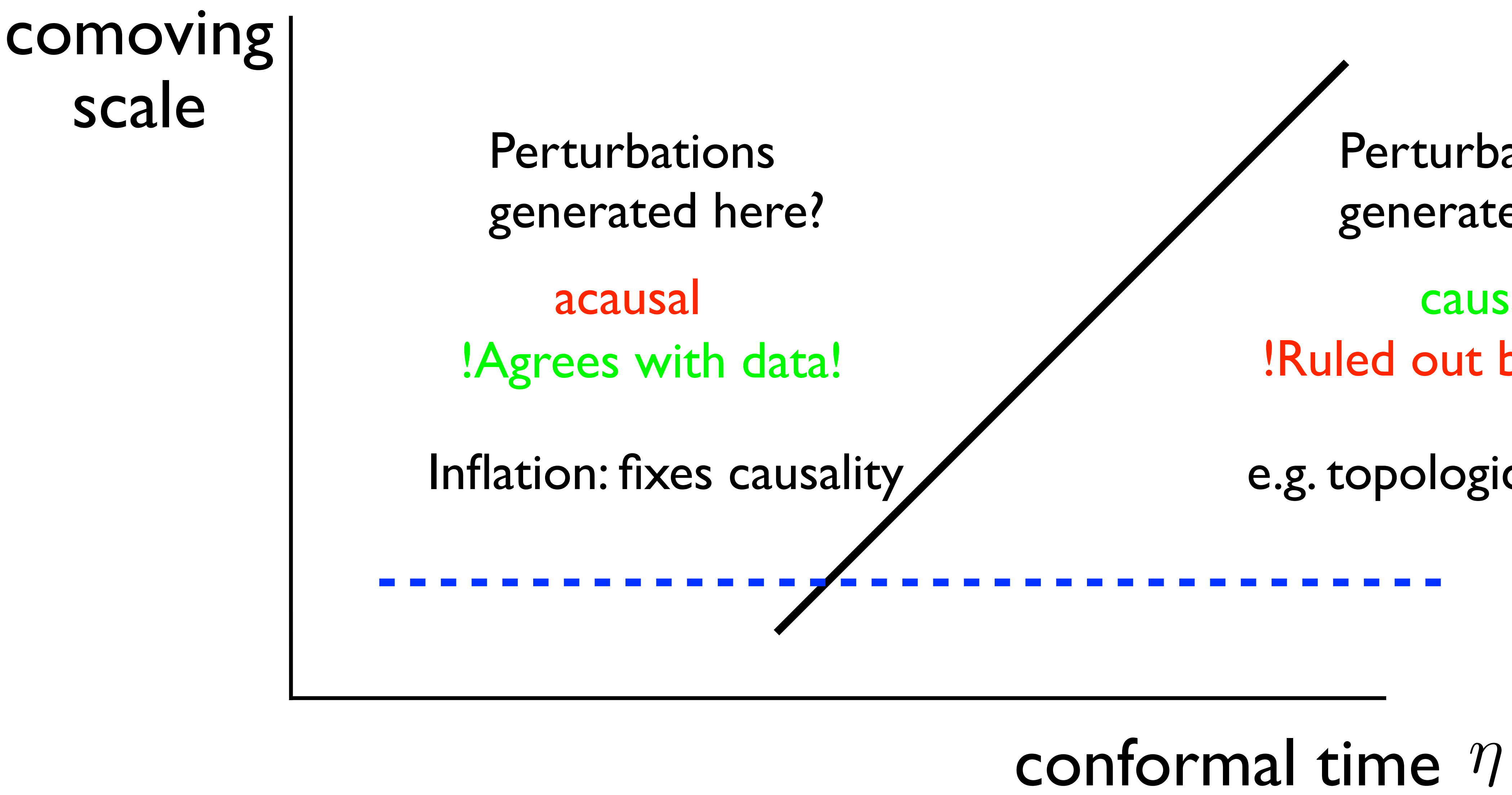

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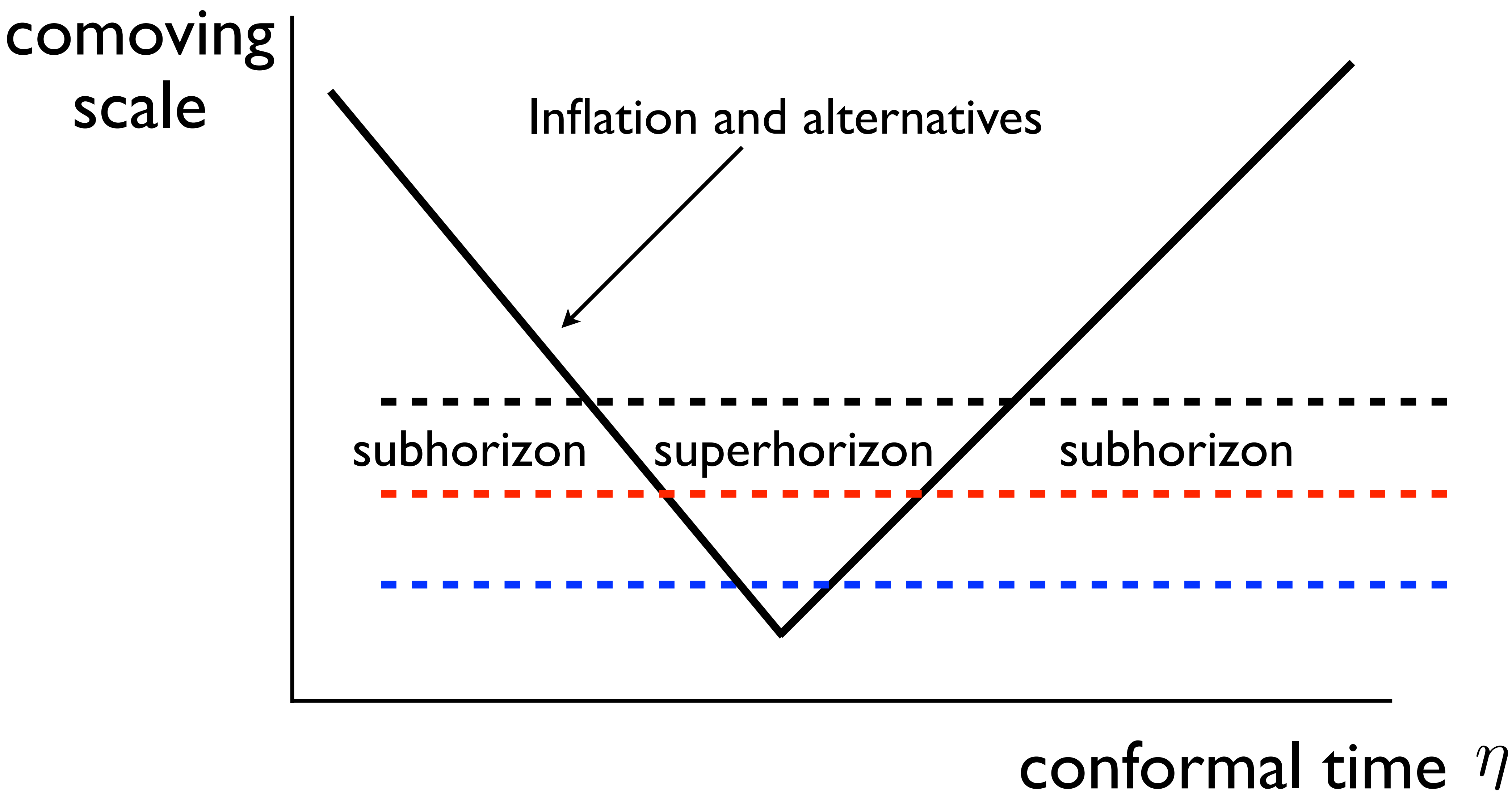

Relativistic Perturbations

- The full model:

$$\delta g_{\mu\nu}(x,t) \qquad \delta\rho_i(x,t) \qquad \delta u^{\mu}(x,t)$$

$$\nabla_{\mu}\delta T^{\mu\nu} = 0 \qquad\qquad \delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu}$$

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- Metric fluctuations

$$\begin{tikzcd}
 & \text{4 scalar} \arrow[r, leftrightarrow] & \delta\rho_i(x,t) \\
\delta g_{\mu\nu}(x,t) \arrow[ur] \arrow[r] \arrow[dr] & \text{4 vector} & \text{decay, not major player} \\
 & \text{2 tensor} & \text{gravity waves}
\end{tikzcd}$$

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- Metric fluctuations

$$\delta g_{\mu\nu}(x,t) \longrightarrow \begin{cases} \text{4 scalar} \longleftrightarrow \delta\rho_i(x,t) \\ \text{4 vector} \quad \text{decay, not major player} \\ \text{2 tensor} \quad \text{gravity waves} \end{cases}$$

- Gauge choice: only 6 DOF.

$$ds^2 = a(\eta)^2 \left[-\left(1+2\Psi\right) d\eta^2 + \left(1+2\Phi\right) \delta_{ij} dx^i dx^j \right]$$

"Conformal Newtonian Gauge"

Important events

Important events

equality

photons tightly
coupled to baryons
imperfect fluid with

$$c_s^2 \simeq 1/2$$

dark matter
perturbations begin
to grow

Important events

equality

recombination

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Important events

equality | recombination

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neutral atoms form

photons and baryons decouple: CMB is released!

baryons begin to collapse into dark matter halos

Important events

equality — recombination — reionization

equality

photons tightly coupled to baryons imperfect fluid with

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recombination

neutral atoms form

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reionization

first non-linear structure

stars reionize the Universe

hierarchical structure formation

some CMB photons re-scatter

Baryon Acoustic Oscillations

- Before recombination, photons and baryons are coupled:

$$\left[\frac{d^2}{dt^2} + 2H\frac{d}{dt} + \left(c_s^2\frac{k_{\rm com}^2}{a^2} - 4\pi G\bar{\rho}\right)\right]\frac{\delta\rho(t,k_{\rm com})}{\bar{\rho}(t)} = 0$$

effective pressure from coupling

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effective pressure from coupling

- Sound horizon:

$$r_s = \int_0^\eta d\eta \; c_s(\eta)$$

$$\frac{\delta\rho}{\bar{\rho}} \propto \cos\left[kr_s(\eta)\right]$$

frequency of oscillation for sound waves

Standard ruler!

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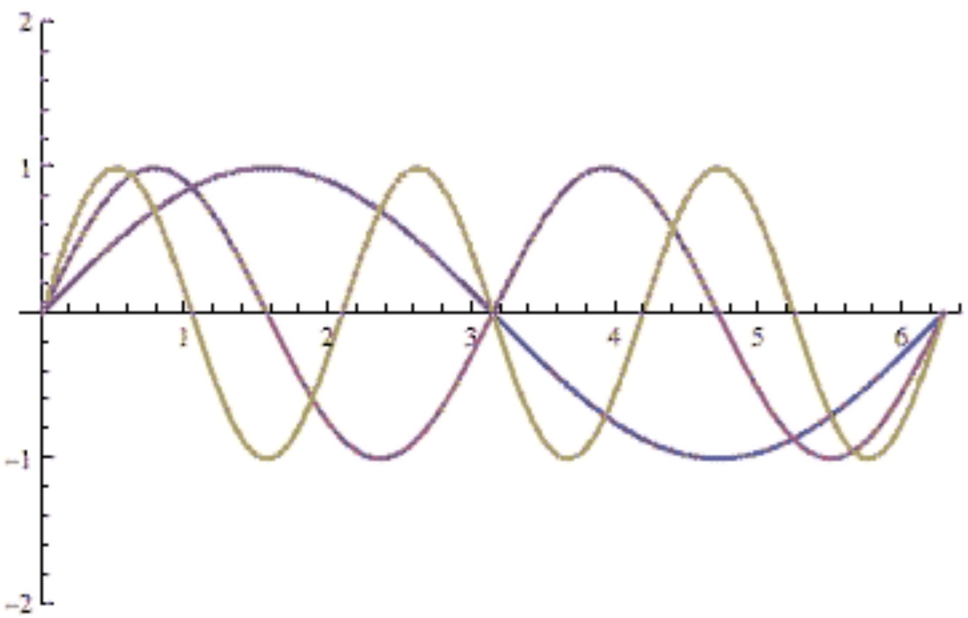
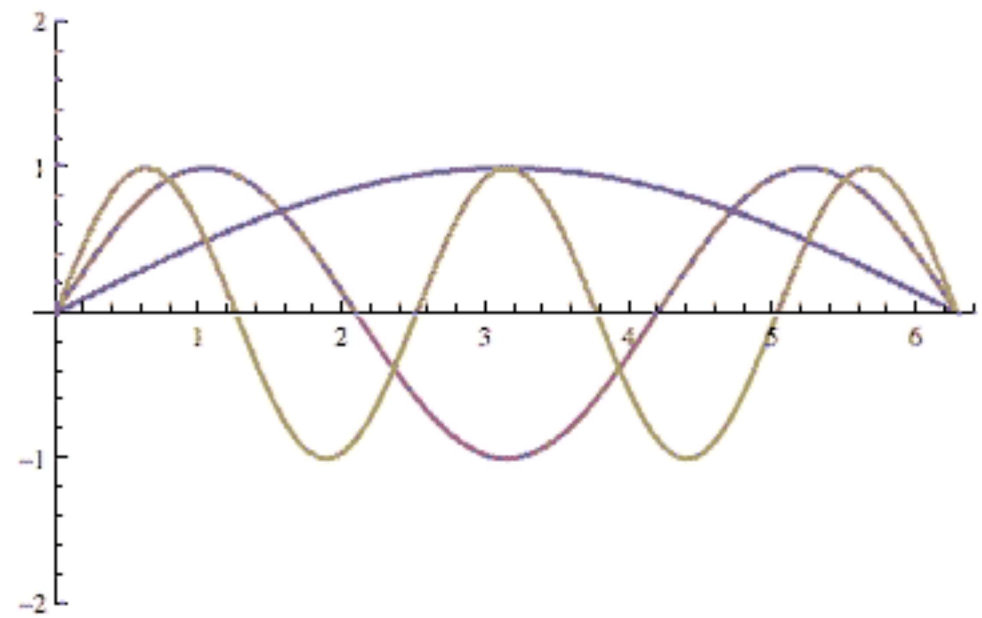


Maximum at recombination



Minimum at recombination

$$k_n = \frac{n\pi}{r_s},\ n = 1,\ 2,\ 3,\ldots$$

The CMB

- The Universe is filled with a gas of photons.

$$dN(t,\vec{x}) = f(t,\vec{x},p)\frac{d^3x d^3p}{(2\pi)^3}$$

$$f = \frac{1}{\exp\left[\frac{p}{T(t)}\right] - 1}$$

Bose-Einstein distribution

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Bose-Einstein distribution

$T(t)$ -- all photons stretched equally

- The Universe's most perfect blackbody.
- The temperature today is 2.73 K.

The CMB

- Perturbations are characterized by:

$$f = \frac{1}{\exp\left[\frac{p}{T(t)(1+\Theta(\vec{x},t,\hat{p}))}\right] - 1} \qquad \Theta = \frac{\delta T}{T}$$

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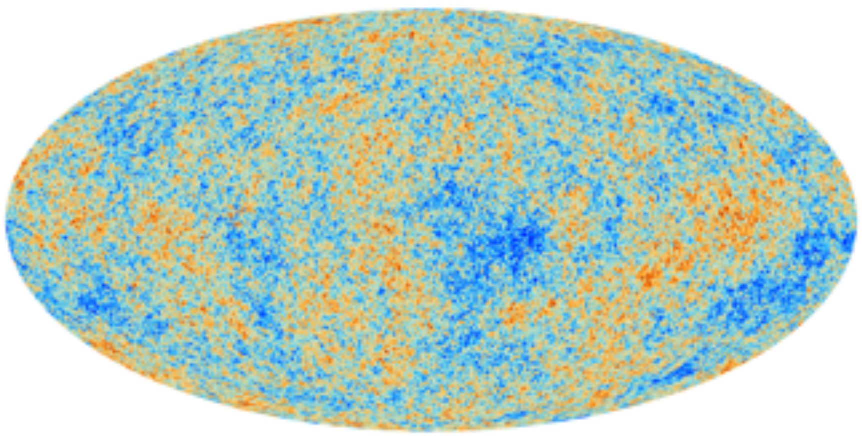
Observers at each position see an anisotropic distribution of photons



$$dN(t_{\rm now}, \vec{x}_{\rm here}) = \int f(t_{\rm now}, \vec{x}, p)\delta(\vec{x}_{\rm here})\frac{d^3x d^3p}{(2\pi)^3}$$

The CMB

Full set of coupled variables go into finding the temperature anisotropy

$$\left(\frac{\delta\rho}{\bar{\rho}}\right)_{\mathrm{dm}},\ v_{\mathrm{dm}},\ \left(\frac{\delta\rho}{\bar{\rho}}\right)_{\mathrm{b}},\ v_{\mathrm{b}},\ \Theta,\ \Phi,\ \Psi$$

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- Sachs-Wolfe -- valid on largest scales

$$\left(\frac{\Delta T}{T}\right)_{\rm fin} = \underbrace{\left(\frac{\Delta T}{T}\right)_{\rm init}}_{\text{Intrinsic temperature variations}} \underbrace{- \Phi_{\rm init}}_{\text{gravitational redshift}} = -\frac{\Phi_{\rm init}}{3}$$

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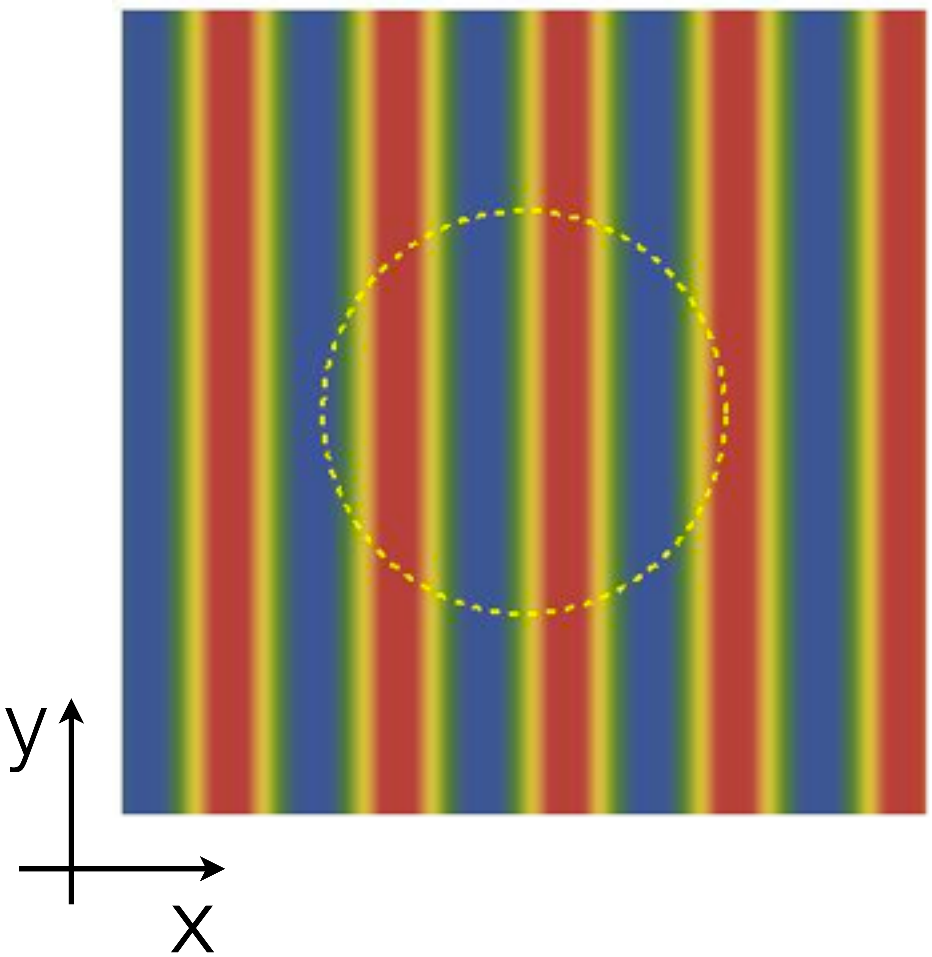
- Integrated Sachs-Wolfe -- time dependence of potentials.

The CMB

- In linear theory, can sum up the contribution from each fourier mode separately:



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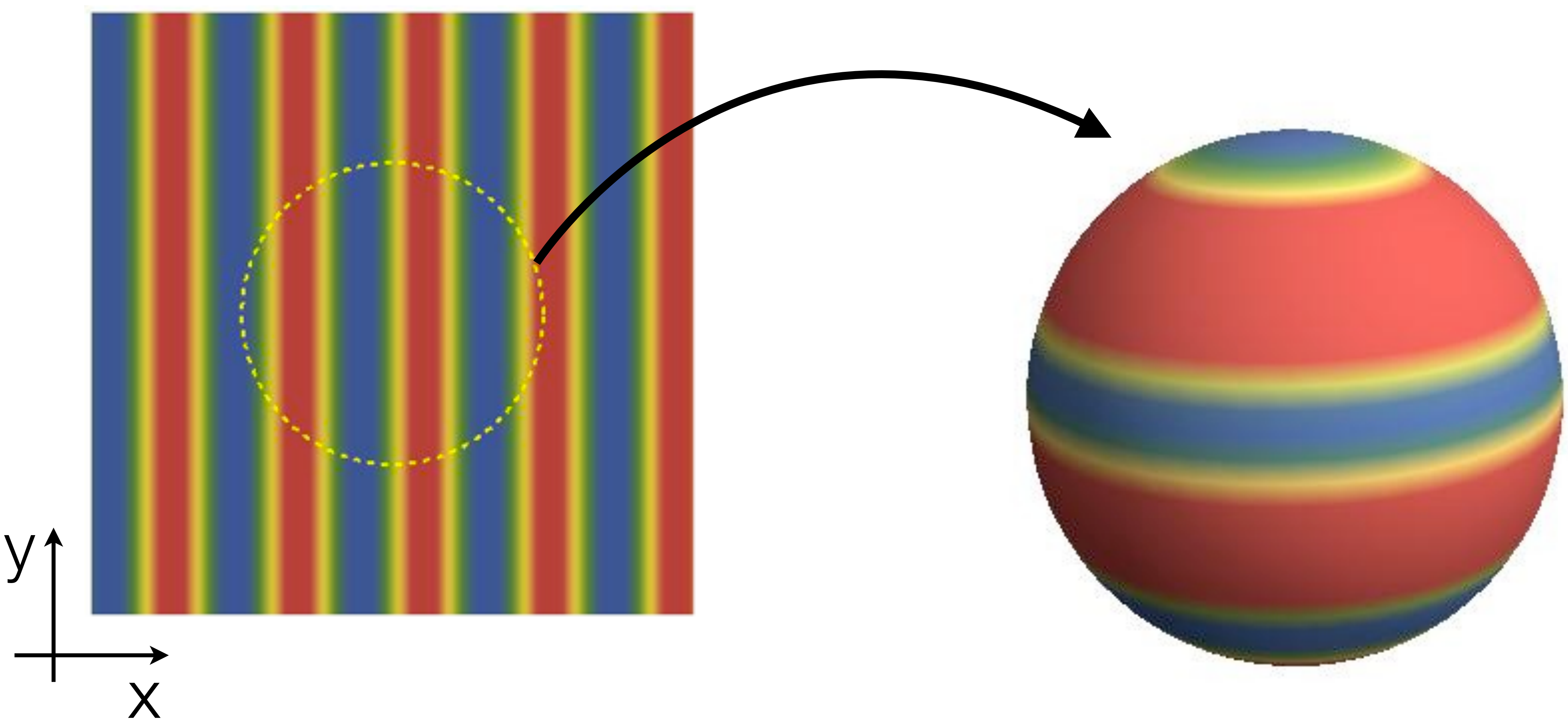

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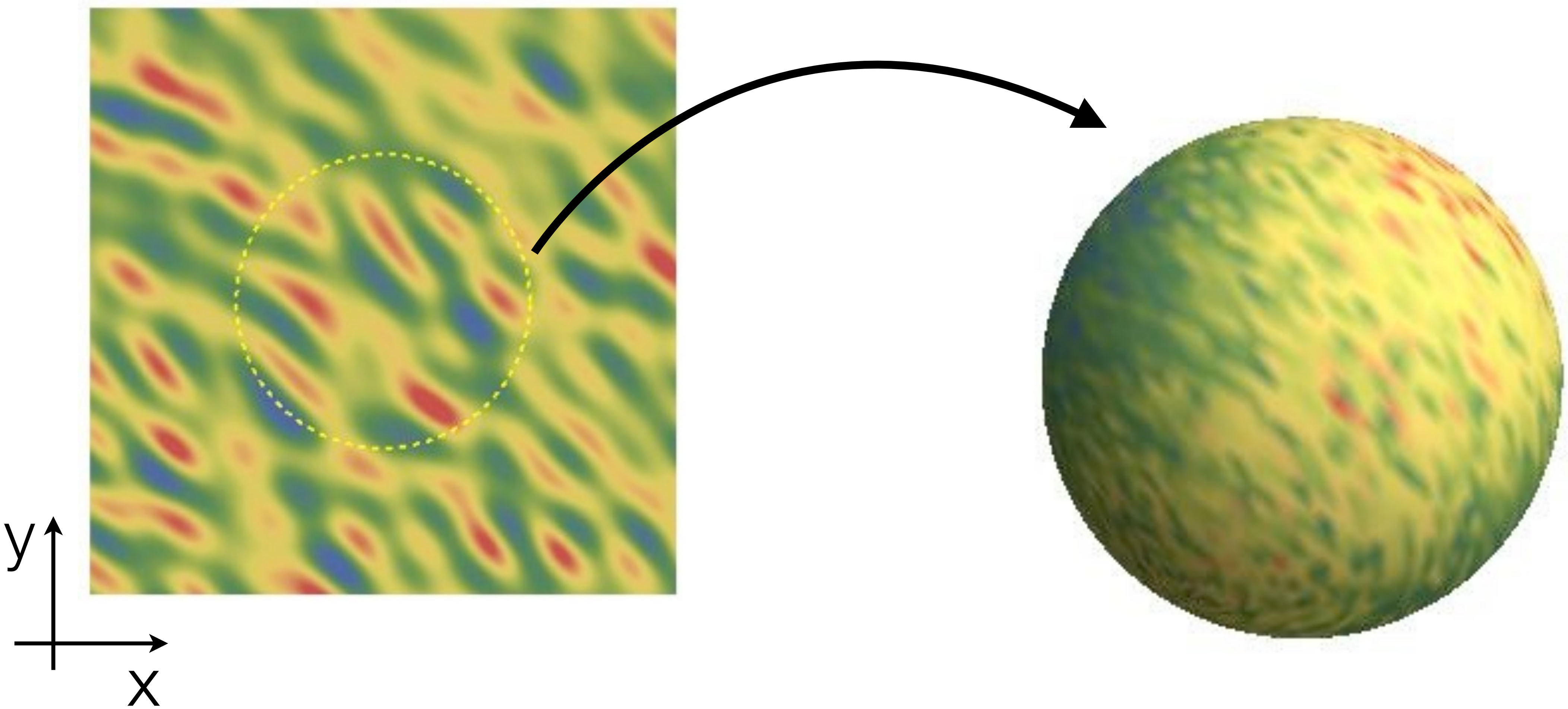

The CMB

- Convenient to perform spherical harmonic transform:

$$\Theta(t, \vec{x}, \hat{p}) = \sum_{\ell} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{p})$$

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$$a_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \Delta_\ell(k) \Phi_{\rm init}(k) Y_{\ell m}(\hat{k})$$

projection, evolution, ISW, etc.

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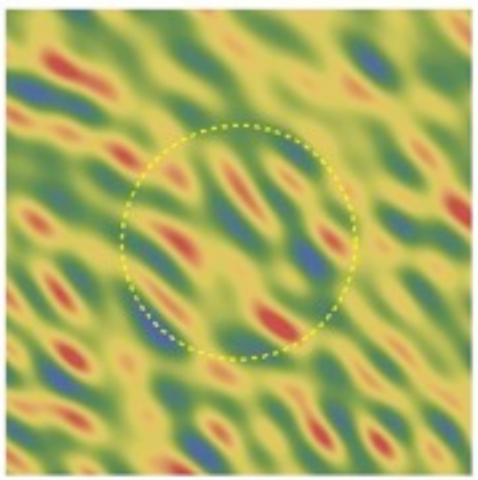
projection, evolution, ISW, etc.

- Computed numerically: CAMB, CMBFast, etc.

The Power Spectrum

- Fluctuations are characterized statistically:



$$\langle \Phi(k)\Phi(k') \rangle = \delta^3(k-k')P(k)$$

$$\langle a_{\ell m} a_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell$$

gaussian:

$$\langle \Phi(k)\Phi(k')\Phi(k'') \rangle = 0$$

....

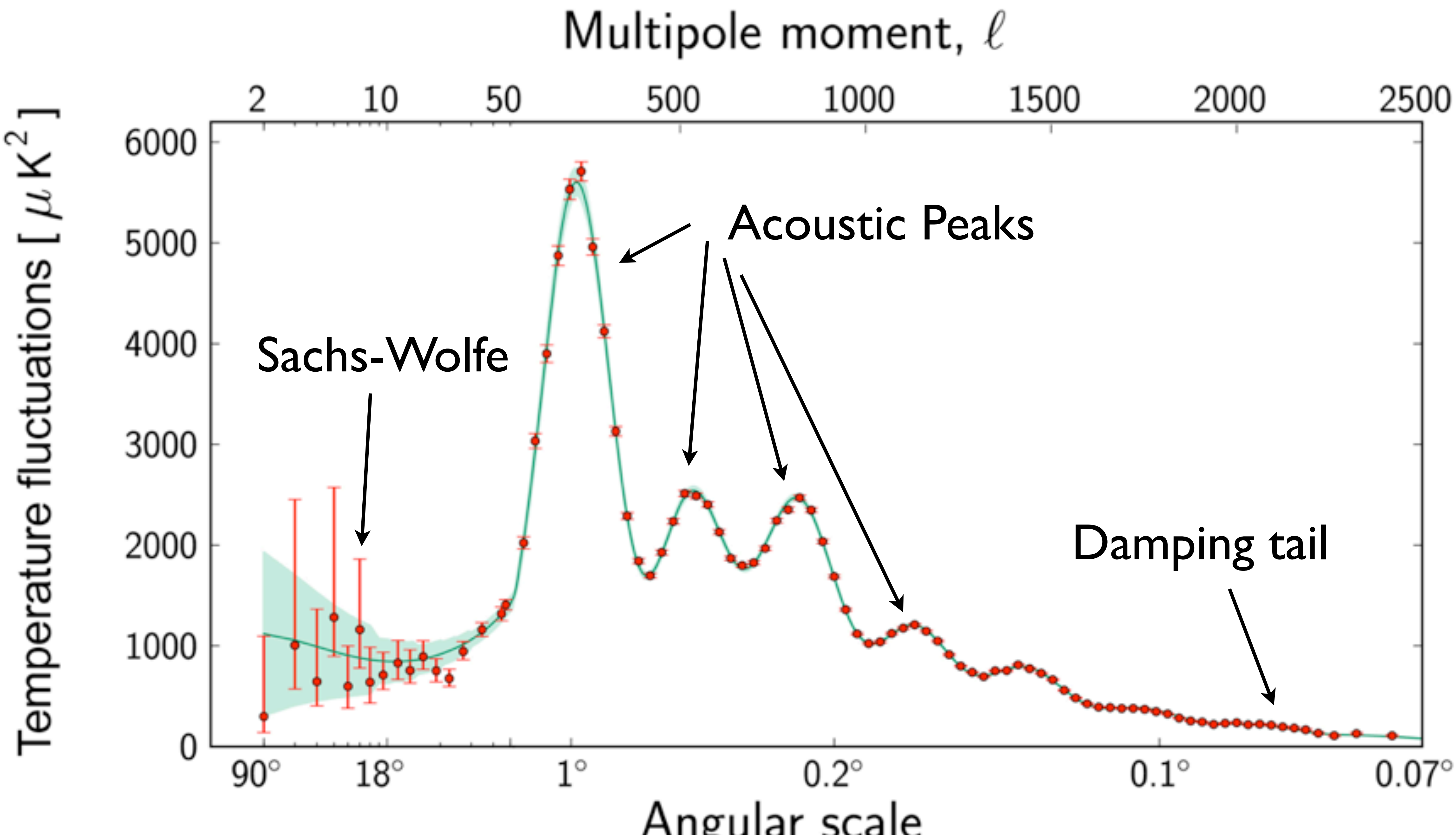
....

$$C_\ell = \frac{2}{\pi}\int dk\ k^2 \Delta_\ell^2(k)P(k)$$

Relates statistics of primordial fluctuations to the statistics of fluctuations in the CMB

The Power Spectrum

$$P(k) = Ak^{n_s - 1} \longrightarrow C_\ell$$



Comments

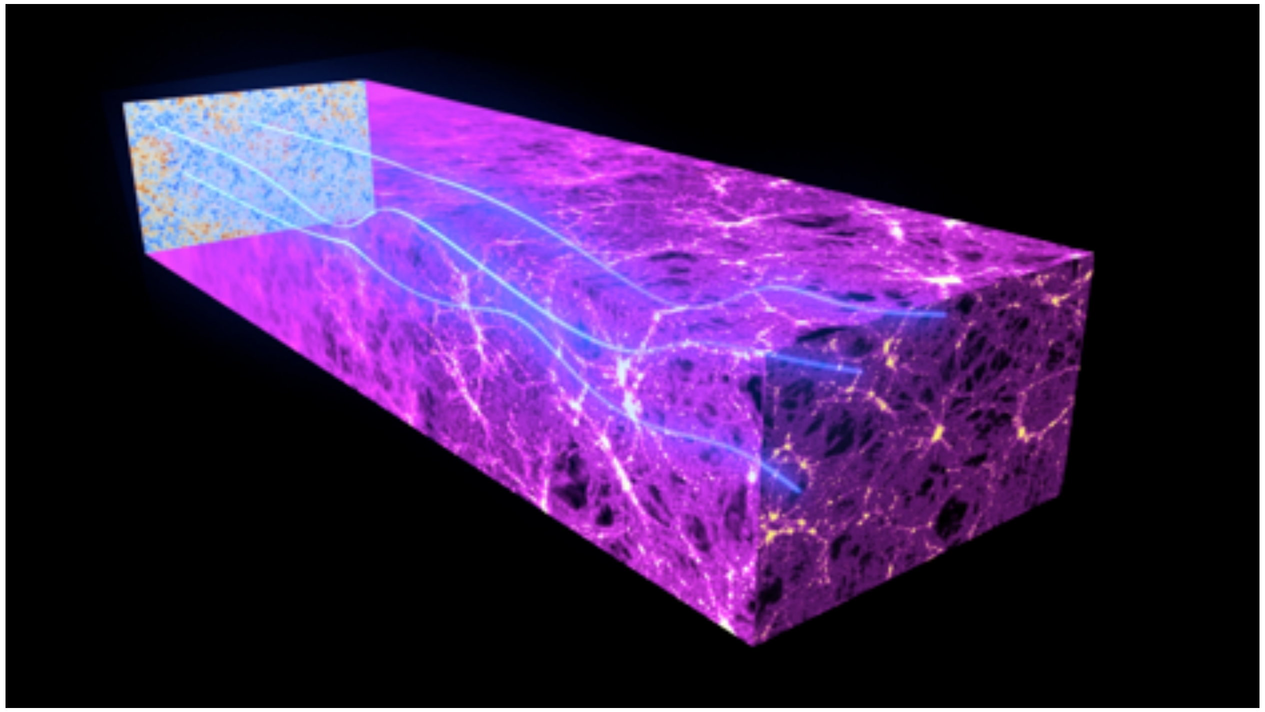
- In a statistically homogeneous and isotropic universe with gaussian fluctuations, the power spectrum is all the information there is.

- There is power on scales of order the size of the observable universe -- superhorizon fluctuations.

- The structure of the acoustic peaks is determined by the contents of the universe as well as the initial conditions.

Other cool things in CMB

- Lensing of the CMB: information on intervening structure



- Polarization of the CMB: primordial gravitational waves
- Sunyaev-Zeldovich effect: shadows of galaxy clusters in the CMB

- Combined mass and number of neutrinos.

6 Parameter Model of the Universe

ΛCDM

	Planck		*Planck*+lensing		*Planck*+WP	
Parameter	Best fit	68% limits	Best fit	68% limits	Best fit	68% limits
$\Omega_b h^2$	0.022068	0.02207 ± 0.00033	0.022242	0.02217 ± 0.00033	0.022032	0.02205 ± 0.00028
$\Omega_c h^2$	0.12029	0.1196 ± 0.0031	0.11805	0.1186 ± 0.0031	0.12038	0.1199 ± 0.0027
$100\theta_{MC}$	1.04122	1.04132 ± 0.00068	1.04150	1.04141 ± 0.00067	1.04119	1.04131 ± 0.00063
τ	0.0925	0.097 ± 0.038	0.0949	0.089 ± 0.032	0.0925	$0.089^{+0.012}_{-0.014}$
n_s	0.9624	0.9616 ± 0.0094	0.9675	0.9635 ± 0.0094	0.9619	0.9603 ± 0.0073
$\ln(10^{10} A_s)$	3.098	3.103 ± 0.072	3.098	3.085 ± 0.057	3.0980	$3.089^{+0.024}_{-0.027}$

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