The Linear Universe

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• Most of cosmology is described by General Relativity and Relativistic Hydrodynamics.
Modelling the Universe

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- The fundamental variables:
  \[ g_{\mu\nu}(x, t), \quad \rho_i(x, t), \quad u^\mu(x, t) \]
  metric \quad fluid densities \quad fluid velocities
Modelling the Universe

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- The fundamental variables:
  \[ g_{\mu\nu}(x, t), \rho_i(x, t), u^\mu(x, t) \]
  metric, fluid densities, fluid velocities
- The laws: Einstein and continuity equations

\[
G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \nabla_\mu T^{\mu\nu} = 0 \quad T_{\mu\nu} = \sum_i (\rho_i + p_i) u_\mu u_\nu + p_i g_{\mu\nu}
\]
Modelling the Universe

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- The fundamental variables:
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  metric \quad fluid densities \quad fluid velocities
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Set of coupled PDE’s -- need initial conditions!
The Linear Universe

- For much of the history of the Universe:
  
  \[ g_{\mu\nu}(x, t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x, t) \]
  
  \[ \rho_i(x, t) = \bar{\rho}_i(t) + \delta \rho_i(x, t) \qquad \delta = \text{small} \]
  
  \[ u^\mu(x, t) = \bar{u}^\mu + \delta u^\mu(x, t) \]
The Linear Universe

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• Why was the universe so nearly homogeneous?
The Linear Universe

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The Linear Universe

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- Why was the universe so nearly homogeneous?

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Universe

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \]

Homogeneous Background

\[ \bar{G}_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu} \]

Fluctuations

\[ \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \]
Initial Conditions

- The homogeneous Universe:

\[ p_i = w_i \rho_i \]

Types of fluids

\[ \bar{\rho}_i(t_0) \]

Densities today

Types of fluids

\[ \bar{\rho}_i(t_0) \]

Densities today
Initial Conditions

- The homogeneous Universe:
  \[ p_i = w_i \rho_i \quad \bar{\rho}_i(t_0) \]
  Types of fluids \hspace{1cm} Densities today

- The Linear Universe:

  Characterize statistics of inhomogeneities!

  \[ \mathcal{P} [g_{\mu\nu} (x, t = 0)] \quad \mathcal{P} [\rho_i (x, t = 0)] \quad \mathcal{P} [u_{i\mu} (x, t = 0)] \]

  Assume our Universe is typical.
6 Parameter Model of the Universe

\( \Lambda \)CDM

3 parameters

2 parameters

1 parameter (for CMB)

\[ \mathcal{P} [\delta g_{\mu\nu} (x, t = 0)] \]
Modelling the universe

Initially small fluctuations collapse to form galaxies, stars, etc.

That’s it!

The rest is details.
• The non-linear Universe

  • GR is highly non-linear - inferring the state of the early universe would be like asking for the weather 100 million years ago based on the weather today.

  • No general classification of metrics - how to characterize initial conditions?

  • Shock waves, singularities, oh my!
Giving Thanks

• The linear Universe

  • Simple evolution allows initial conditions to be inferred.
  
  • Background evolution and growth of structure can be analyzed separately.
  
  • Simple classification of initial conditions and metric degrees of freedom.
  
  • Physics on different scales evolves independently (Fourier modes independent).
The rest

Now for some details....
The homogeneous universe
The homogeneous universe

13.7 Billion Years: the present.
9.1 Billion Years: our sun ignites.
100 million years: galaxies and first stars form.
380,000 years: neutral atoms form.
1 second: atomic nuclei form.
$10^{-6}$ seconds: protons and neutrons form.

?Big Bang?
The homogeneous universe

- The metric in a flat, homogeneous, isotropic universe:

\[ ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \]
The homogeneous universe

- The metric in a flat, homogeneous, isotropic universe:

\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j \]

comoving coordinates

\[ \Delta x \]

\[ \Delta s \]

constant comoving distance = growing physical distance
The homogeneous universe

• Conformal time:

\[ \eta = \int \frac{dt}{a(t)} \]

\[ ds^2 = a^2(\eta) \left[ -d\eta^2 + \delta_{ij} dx^i dx^j \right] \]
The homogeneous universe

- Conformal time:
  \[ \eta = \int \frac{dt}{a(t)} \]
  \[ ds^2 = a^2(\eta) \left[ -d\eta^2 + \delta_{ij} dx^i dx^j \right] \]

\[ \eta - \eta_0 = \pm (x - x_0) \]

The big bang: when \( a(t) = 0 \)
The homogeneous universe

- Conformal time:
  \[ \eta = \int \frac{dt}{a(t)} \]
  \[ ds^2 = a^2(\eta) \left[-d\eta^2 + \delta_{ij} dx^i dx^j \right] \]

The big bang: when \( a(t) = 0 \)

Particle horizon

\[ \Delta x = \int \frac{dt}{a(t)} = \int_{a=0}^{a_0=1} \frac{d\ln(a)}{aH} \]

Comoving horizon
The homogeneous universe

- Equations of motion in a homogeneous universe:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \rightarrow \quad H^2 \equiv \left( \frac{\dot{a}}{a} \right) = \frac{8\pi G \rho}{3} \]
The homogeneous universe

- Equations of motion in a homogeneous universe:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \Rightarrow \quad H^2 = \left( \frac{\dot{a}}{a} \right) = \frac{8\pi G \rho}{3} \]

\[ \nabla_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{\rho} = -3H (\rho + p) \]

\[ p = w\rho \]
The homogeneous universe

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\[ \nabla_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{\rho} = -3H (\rho + p) \]

\[ p = w \rho \]

- Solutions:

\[ \rho = \rho_0 a^{-3(1+w)} \quad a(t) = a_0 t^{\frac{2}{3(1+w)}} \]

different fluids gravitate differently!
The homogeneous universe

\[ \rho = \rho_0 a^{-3(1+w)} \]

radiation \( w = \frac{1}{3}, \rho \propto a^{-4} \)

matter \( w = 0, \rho \propto a^{-3} \)

dark energy \( w = -1, \rho \propto \text{const.} \)
The homogeneous universe

- Evolution of the scale factor:

\[ a(t) \]
The homogeneous universe

• Energy budget:

\[
\left( \frac{H}{H_0} \right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} \quad \sum_i \Omega_i = 1
\]
The homogeneous universe

- Energy budget:

\[
\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} \quad \sum_i \Omega_i = 1
\]

\[
\Omega_r \sim 10^{-4}
\]
The homogeneous universe

- Energy budget:

\[
\left( \frac{H}{H_0} \right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} \quad \sum_i \Omega_i = 1
\]

\[H_0 = 100h \frac{\text{km}}{\text{s Mpc}} \]

\[
= \frac{h}{3000 \text{ Mpc}}
\]

\[1 \text{ Mpc} = 3 \times 10^{23} \text{ meters}\]

\[1 \text{ Mpc} = 3.3 \times 10^6 \text{ Lyr}\]

\[\Omega_r \sim 10^{-4}\]
The homogeneous universe

- Redshift: \[ z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{obs}}} \]

\[ a = \frac{1}{1 + z} \]

\[ a = 1, \quad z = 0 \]
The homogeneous universe

- Redshift: \( z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{obs}}} \)

\[
a = \frac{1}{1 + z}
\]

\[
a = 1, \quad z = 0
\]

\[
\begin{align*}
\zeta_{\text{eq}} &= 3400, \\
\zeta_* &= 1090, \\
\zeta_{\text{re}} &\approx 11, \\
\zeta_{\text{gal}} &\approx 11 - 12, \\
\zeta_{\text{surveys}} &\ll 1, \\
\zeta_{\Lambda} &= .28, \\
\zeta_{\text{Virgo}} &= .003
\end{align*}
\]
The Inhomogeneous Universe

- There is structure in the Universe:

\[ \omega = \frac{1}{3}, \quad \propto \frac{1}{a^4} (32) \]

\[ \omega = 0, \quad \propto \frac{1}{a^3} (33) \]

\[ \omega = 1, \quad \propto \text{const.} (34) \]

\[ \Delta r \sim 10^{10} (35) \]

\[ H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} (36) \]

\[ 1 \text{ Mpc} = 3 \times 10^6 \text{ Lyr} (37) \]

- In Fourier space:

\[ \rho(x, t) = \frac{1}{V} \sum \rho(k, t) e^{i\vec{k} \cdot \vec{x}} \]

\[ \rho(k, t) \]

\[ \bar{\rho}(t) \]

0 \quad k
The Inhomogeneous Universe

- There is structure in the Universe:

\[
\begin{align*}
\frac{\omega}{1} = & \frac{1}{3}, \quad \frac{\omega}{a^4} (32) \\
0 = & \frac{1}{3}, \quad \frac{\omega}{a^3} (33) \\
1 = & \text{const.} (34)
\end{align*}
\]

- In Fourier space:

\[
\rho(x, t) = \frac{1}{V} \sum \rho(k, t) e^{ik \cdot \vec{x}}
\]

\[
\rho(k, t)
\]

\[
\tilde{\rho}(t)
\]

\[
0 \quad k
\]
There is structure in the Universe:

\[ w = \frac{1}{3}, \quad \frac{\dot{a}}{a^4} \] (32)

\[ w = 0, \quad \frac{\dot{a}}{a^3} \] (33)

\[ w = 1, \quad \frac{\dot{a}}{\text{const}}. \] (34)

\[ \bar{r} \ll 10^{-4} \] (35)

\[ H_0 = 100h \text{ km s}^{-1} \text{Mpc}^{-1} \] (36)

\[ 1 \text{ Mpc} = 3 \times 10^6 \text{Lyr} \] (38)

\[ \rho(x, t) = \frac{1}{V} \sum \rho(k, t) e^{i \vec{k} \cdot \vec{x}} \] (39)

\[ \rho(k, t) \]

\[ \bar{\rho}(t) \]

\[ 0 \]

\[ k \]
There is structure in the Universe:

- In Fourier space:
  \[ \rho(x, t) = \frac{1}{V} \sum \rho(k, t) e^{i\mathbf{k} \cdot \mathbf{x}} \]

- Large on small scales
The Inhomogeneous Universe

• There is structure in the Universe:

Structure: \( \frac{\delta \rho}{\bar{\rho}} \gtrsim 1 \)

\( \mathcal{O}(10^6) \) galaxies
\( \mathcal{O}(10^3) \) clusters
\( \mathcal{O}(1) \) superclusters
The Inhomogeneous Universe

- There is structure in the Universe:

  Structure: \[ \frac{\delta \rho}{\bar{\rho}} \gtrsim 1 \]

  \( O(10^6) \) galaxies

  \( O(10^3) \) clusters

  \( O(1) \) superclusters

- There is structure on all scales which have had a chance to undergo gravitational collapse.

- The largest structures in the Universe define a scale above which the fluctuations in density are linear:

  \[ k = 0.1 - 0.01 \text{Mpc}^{-1} \]

  \( \lambda \sim 1\% \) of observable Universe wave number
The Inhomogeneous Universe

There is structure in the Universe:

Luminous:
- Baryons
- Photons

Semi-Luminous:
- Neutrinos

Thursday, 4 July, 13
The Inhomogeneous Universe

- There is structure in the Universe:
  - Luminous: Baryons, Photons
  - Semi-Luminous: Neutrinos
  - Dark: Dark Matter, Dark Energy
Gravitational Instability

- For non-relativistic matter in flat space:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]
 continuity

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} + \frac{\nabla p}{\rho} + \nabla \phi = 0
\]
 Euler

\[
\nabla^2 \phi = 4\pi G \rho
\]
 Poisson
Gravitational Instability

- For non-relativistic matter in flat space:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0 \\
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} + \frac{\nabla p}{\rho} + \nabla \phi &= 0
\end{align*}
\]

\text{Continuity} \quad \text{Euler}

\nabla^2 \phi = 4\pi G \rho

\text{Poisson}

- Linearize:

\[
\begin{align*}
\rho &= \bar{\rho} + \delta \rho \\
\phi &= \bar{\phi} + \delta \phi \\
\vec{v} &= \bar{\vec{v}} + \delta \vec{v}
\end{align*}
\]

\[p = \bar{p} + c_s^2 \delta \rho\]

\[c_s^2 = \frac{\partial p}{\partial \rho} = w\]
Gravitational Instability

- Linearized equation of motion:

\[
\frac{\partial^2 \delta \rho}{\partial t^2} - c_s^2 \nabla^2 \delta \rho - 4\pi G \bar{\rho} \delta \rho = 0
\]
Gravitational Instability

- Linearized equation of motion:

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\frac{\partial^2 \delta \rho}{\partial t^2} - c_s^2 \nabla^2 \delta \rho - 4\pi G \bar{\rho} \delta \rho = 0
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- Fourier transform:

\[
\delta \rho(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \delta \rho(t, k) e^{i\vec{k} \cdot \vec{x}}
\]
Gravitational Instability

• Linearized equation of motion:

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Gravitational Instability

• Linearized equation of motion:

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\]

\[
\delta \rho(t, k) = A \exp (i \omega(k)t) + B \exp (-i \omega(k)t)
\]

\[
\omega(k) = \sqrt{k^2 c_s^2 - 4\pi G \bar{\rho}}
\]
Gravitational Instability

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Gravitational Instability

\[ \omega(k) = \sqrt{k^2 c_s^2 - 4\pi G \bar{\rho}} \]

- Jeans scale: competition between pressure and gravity

\[ \lambda_J = \frac{2\pi}{k_J} = c_s \left( \frac{\pi}{G \bar{\rho}} \right)^{1/2} \]
Gravitational Instability

\[ \omega(k) = \sqrt{k^2 c_s^2 - 4\pi G \bar{\rho}} \]

- Jeans scale: competition between pressure and gravity

\[ \lambda_J = \frac{2\pi}{k_J} = c_s \left( \frac{\pi}{G \bar{\rho}} \right)^{1/2} \]

\[ \lambda < \lambda_J \]

sound waves

\[ \lambda > \lambda_J \]

gravitational collapse
Gravitational Instability

- In an expanding universe waves are stretched:

\[ \frac{\partial^2 \phi}{\partial t^2} - c^2 s^{4\pi G \bar{\rho}} = 0 \]  
\[ \frac{\partial^2 \phi}{\partial t^2} - c^2 s^{4\pi G \bar{T}} = 0 \]  
\[ \phi(t, k) = A \exp \left( i k \cdot x \right) + B \exp \left( -i k \cdot x \right) \]  
\[ \lambda_{ph} = a(t) \lambda_{com} \]
Gravitational Instability

- In an expanding universe waves are stretched:

\[ \lambda_{ph} = a(t) \lambda_{com} \]

- Only gravitationally bound (non-linear) structures separate from the Hubble flow.

- Expansion inhibits collapse.

\[ \Delta x \rightarrow \Delta s \]

\[ \uparrow \text{time} \]
Gravitational Instability

\[ t_{\text{coll}} \sim (4\pi G \bar{\rho})^{-1/2} \sim H^{-1} \]

Expansion will be relevant!
Gravitational Instability

\[ t_{\text{coll}} \sim (4\pi G \bar{\rho})^{-1/2} \sim H^{-1} \]

Expansion will be relevant!

- Including expansion (on small scales):

\[
\left[ \frac{d^2}{dt^2} + 2H \frac{d}{dt} + \left( c_s^2 \frac{k_{\text{com}}^2}{a^2} - 4\pi G \bar{\rho} \right) \right] \frac{\delta \rho(t, k_{\text{com}})}{\bar{\rho}(t)} = 0
\]
Gravitational Instability

\[ t_{\text{coll}} \sim (4\pi G \bar{\rho})^{-1/2} \sim H^{-1} \]

Expansion will be relevant!

- Including expansion (on small scales):

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\]

\[ c_s^2 = 0 \]

<table>
<thead>
<tr>
<th>Radiation</th>
<th>Matter</th>
<th>Dark Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\delta \rho}{\bar{\rho}} \propto \log(a) )</td>
<td>( \frac{\delta \rho}{\bar{\rho}} \propto a )</td>
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Gravitational Instability

\[ t_{\text{coll}} \sim (4\pi G \bar{\rho})^{-1/2} \sim H^{-1} \quad \text{Expansion will be relevant!} \]

- Including expansion (on small scales):

\[ \left[ \frac{d^2}{dt^2} + 2H \frac{d}{dt} + \left( c_s^2 \frac{k_{\text{com}}^2}{a^2} - 4\pi G \bar{\rho} \right) \right] \frac{\delta \rho(t, k_{\text{com}})}{\bar{\rho}(t)} = 0 \]

\[ c_s^2 = 0 \quad \text{radiation} \quad \text{matter} \quad \text{dark energy} \]

\[ \frac{\delta \rho}{\bar{\rho}} \propto \log(a) \quad \frac{\delta \rho}{\bar{\rho}} \propto a \quad \frac{\delta \rho}{\bar{\rho}} \propto \text{const.} \]

**To have structure, need:**

\[ \frac{\delta \rho}{\bar{\rho}}(t_{\text{eq}}) \gtrsim 3 \times 10^{-4} \]
Gravitational Instability

- An important scale: comoving horizon \( \frac{1}{aH} \)
- Horizon crossing: \( k = aH \)

**Diagram:**
- Comoving scale
- Conformal time \( \eta \)
- Horizon crossing points
- Superhorizon
- Subhorizon

Thursday, 4 July, 13
Gravitational Instability

- An important scale: comoving horizon \( \frac{1}{aH} \)
- Horizon crossing: \( k = aH \)

\[
\begin{array}{c}
\text{comoving scale} \\
\hline
\text{horizon crossing: } \frac{1}{aH} = \eta \\
\text{no gravitational collapse} \\
\text{GR necessary} \\
\text{gravitational collapse} \\
\text{conformal time } \eta
\end{array}
\]
Gravitational Instability

- An important scale: comoving horizon $\frac{1}{aH}$
- Horizon crossing: $k = aH$

Diagram:

- Comoving scale
- Conformal time $\eta$
- Smaller scales collapse first
Gravitational Instability

- An important scale: comoving horizon \( \frac{1}{aH} \)
- Horizon crossing: \( k = aH \)

extrapolating, all scales start out superhorizon

comoving scale

conformal time \( \eta \)
Gravitational Instability

- An important scale: comoving horizon \( \frac{1}{aH} \)
- Horizon crossing: \( k = aH \)
Gravitational Instability

- An important scale: comoving horizon $\frac{1}{aH}$
- Horizon crossing: $k = aH$

Perturbations generated here? causal
!Ruled out by data!
e.g. topological defects

comoving scale

conformal time $\eta$
Gravitational Instability

- An important scale: comoving horizon $\frac{1}{aH}$
- Horizon crossing: $k = aH$

Perturbations generated here?
- acausal
  - !Agrees with data!

Inflation: fixes causality

Perturbations generated here?
- causal
  - !Ruled out by data!

e.g. topological defects

comoving scale

conformal time $\eta$
Gravitational Instability

- An important scale: comoving horizon $\frac{1}{aH}$
- Horizon crossing: $k = aH$

Diagram:
- Comoving scale
- Inflation and alternatives
- Subhorizon
- Superhorizon
- Conformal time $\eta$
Relativistic Perturbations

- The full model:
  \[ \delta g_{\mu\nu}(x, t) \quad \delta \rho_i(x, t) \quad \delta u^\mu(x, t) \]
  \[ \nabla_\mu \delta T^{\mu\nu} = 0 \quad \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \]
Relativistic Perturbations

- The full model:
  \[
  \delta g_{\mu\nu}(x, t) \quad \delta \rho_i(x, t) \quad \delta u^\mu(x, t)
  \]
  \[
  \nabla_\mu \delta T^{\mu\nu} = 0 \quad \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}
  \]

- Metric fluctuations
  
  \[
  \delta g_{\mu\nu}(x, t) \quad \rightarrow \quad 4 \text{ scalar} \quad \leftrightarrow \quad \delta \rho_i(x, t)
  \]
  
  4 scalar \quad \rightarrow \quad 4 \text{ vector} \quad \text{decay, not major player}

  \[
  \rightarrow \quad 2 \text{ tensor} \quad \text{gravity waves}
  \]
Relativistic Perturbations

- The full model:
  \[ \delta g_{\mu\nu}(x, t) \quad \delta \rho_i(x, t) \quad \delta u^\mu(x, t) \]
  \[ \nabla_\mu \delta T^{\mu\nu} = 0 \quad \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \]

- Metric fluctuations
  \[ \delta g_{\mu\nu}(x, t) \]
  - 4 scalar \[\delta \rho_i(x, t)\]
  - 4 vector \[\text{decay, not major player}\]
  - 2 tensor \[\text{gravity waves}\]

- Gauge choice: only 6 DOF.

\[ ds^2 = a(\eta)^2 \left[ - (1 + 2\Psi) \, d\eta^2 + (1 + 2\Phi) \, \delta_{ij} \, dx^i \, dx^j \right] \]

"Conformal Newtonian Gauge"
Important events

equality
Important events

equality

photons tightly coupled to baryons
imperfect fluid with

\[ c_s^2 \approx 1/2 \]

dark matter
perturbations begin to grow

\[ \Theta(10^6) (10^3) (1) (10^5) \]
Important events

<table>
<thead>
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<th>equality</th>
<th>recombination</th>
</tr>
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$\tau_{eq} \gg 3 \times 10^4 (100)$

$\tau_{eq} \gg 1 \times 10^1 (101)$

$O(10^6) (102)$

$O(10^3) (103)$

$O(1) (104)$

$\sim 1\%$ of observable Universe (105)
Important events

- **Equality**
  - Photons tightly coupled to baryons
  - Imperfect fluid with
    \[ c_s^2 \approx 1/2 \]
  - Dark matter perturbations begin to grow

- **Recombination**
  - Neutral atoms form
  - Photons and baryons decouple:
    - CMB is released!
  - Baryons begin to collapse into dark matter halos
Important events

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## Important events

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<td>hierarchical structure formation</td>
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<td></td>
<td>some CMB photons re-scatter</td>
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Baryon Acoustic Oscillations

• Before recombination, photons and baryons are coupled:

\[
\left[ \frac{d^2}{dt^2} + 2H \frac{d}{dt} + \left( c_s^2 \frac{k_{\text{com}}^2}{a^2} - 4\pi G \bar{\rho} \right) \right] \frac{\delta \rho(t, k_{\text{com}})}{\bar{\rho}(t)} = 0
\]

effective pressure from coupling
Baryon Acoustic Oscillations

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\]

\( r_s = \int_0^\eta d\eta \ c_s(\eta) \)

\( \frac{\delta \rho}{\bar{\rho}} \propto \cos [kr_s(\eta)] \)

- Sound horizon:

  \( r_s = \int_0^\eta d\eta \ c_s(\eta) \)

  frequency of oscillation for sound waves

  Standard ruler!
Before recombination, photons and baryons are coupled:

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\left[ \frac{d^2}{dt^2} + 2H \frac{d}{dt} + \left( c_s^2 \frac{k_{\text{com}}^2}{a^2} - 4\pi G \bar{\rho} \right) \right] \frac{\delta \rho(t, k_{\text{com}})}{\bar{\rho}(t)} = 0
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effective pressure from coupling

Maximum at recombination

Minimum at recombination

\[
k_n = \frac{n\pi}{r_s}, \ n = 1, \ 2, \ 3, \ldots
\]

\[\text{Maximum at recombination}\]

\[\text{Minimum at recombination}\]
The CMB

- The Universe is filled with a gas of photons.

\[
\frac{dN(t, \vec{x})}{d^3x d^3p} = f(t, \vec{x}, p) \frac{d^3x d^3p}{(2\pi)^3}
\]

\[
f = \frac{1}{\exp \left[ \frac{p}{T(t)} \right] - 1}
\]

Bose-Einstein distribution
The CMB

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\]

Bose-Einstein distribution

\[
T(t) \quad \text{-- all photons stretched equally}
\]

- The Universe’s most perfect blackbody.
- The temperature today is 2.73 K.
The CMB

- Perturbations are characterized by:

\[
 f = \frac{1}{\exp \left[ \frac{p}{T(t)(1+\Theta(x,t,\hat{p}))} \right] - 1} \quad \Theta = \frac{\delta T}{T}
\]
The CMB

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\]

\[
\Theta = \frac{\delta T}{T}
\]

Observers at each position see an anisotropic distribution of photons

\[
dN(t_{\text{now}}, \vec{x}_{\text{here}}) = \int f(t_{\text{now}}, \vec{x}, p) \delta(\vec{x}_{\text{here}}) \frac{d^3 x d^3 p}{(2\pi)^3}
\]
The CMB

Full set of coupled variables go into finding the temperature anisotropy

\[
\left( \frac{\delta \rho}{\bar{\rho}} \right)_{\text{dm}}, \nu_{\text{dm}}, \left( \frac{\delta \rho}{\bar{\rho}} \right)_{\text{b}}, \nu_{\text{b}}, \Theta, \Phi, \Psi
\]
The CMB

Full set of coupled variables go into finding the temperature anisotropy

\begin{align*}
\left( \frac{\delta \rho}{\bar{\rho}} \right)_{\text{dm}}, \quad v_{\text{dm}}, \quad \left( \frac{\delta \rho}{\bar{\rho}} \right)_{\text{b}}, \quad v_{\text{b}}, \quad \Theta, \quad \Phi, \quad \Psi
\end{align*}

- Sachs-Wolfe -- valid on largest scales

\begin{align*}
\left( \frac{\Delta T}{T} \right)_{\text{fin}} &= \left( \frac{\Delta T}{T} \right)_{\text{init}} - \Phi_{\text{init}} = -\frac{\Phi_{\text{init}}}{3}
\end{align*}

Intrinsic temperature variations

gravitational redshift
The CMB

Full set of coupled variables go into finding the temperature anisotropy

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\]

Intrinsic temperature variations

Gravitational redshift

- Integrated Sachs-Wolfe -- time dependence of potentials.
The CMB

- In linear theory, can sum up the contribution from each fourier mode separately:
The CMB

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The CMB

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The CMB

- Convenient to perform spherical harmonic transform:

\[
\Theta(t, \vec{x}, \hat{p}) = \sum_\ell \sum_{m=-\ell}^\ell a_{\ell m} Y_{\ell m}(\hat{p})
\]
The CMB

- Convenient to perform spherical harmonic transform:

\[ \Theta(t, \vec{x}, \hat{p}) = \sum_{\ell} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{p}) \]

- The transfer function: \( \Phi_{\text{init}}(k) \rightarrow a_{\ell m} \)

\[ a_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} \Delta_{\ell}(k) \Phi_{\text{init}}(k) Y_{\ell m}(\hat{k}) \]

projection, evolution, ISW, etc.
The CMB

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\]

projection, evolution, ISW, etc.

• Computed numerically: CAMB, CMBFast, etc.
The Power Spectrum

- Fluctuations are characterized statistically:

\[
\langle \Phi(k)\Phi(k') \rangle = \delta^3(k-k')P(k)
\]

\[
\left\langle \Phi(k)\Phi(k')\Phi(k'') \right\rangle = 0
\]

\[
\langle a_{\ell m}a_{\ell'm'} \rangle = \delta_{\ell\ell'}\delta_{mm'}C_{\ell}
\]

\[
C_{\ell} = \frac{2}{\pi} \int dk \; k^2 \Delta_\ell^2(k)P(k)
\]

Relates statistics of primordial fluctuations to the statistics of fluctuations in the CMB
The Power Spectrum

\[ P(k) = A k^{n_s - 1} \rightarrow C_\ell \]

\[ \text{Temperature fluctuations} \ [\mu K^2] \]

\[ \text{Angular scale} \]

\[ \text{Multipole moment, } \ell \]

Acoustic Peaks

Sachs-Wolfe

Damping tail
In a statistically homogeneous and isotropic universe with gaussian fluctuations, the power spectrum is all the information there is.

There is power on scales of order the size of the observable universe -- superhorizon fluctuations.

The structure of the acoustic peaks is determined by the contents of the universe as well as the initial conditions.
Other cool things in CMB

- Lensing of the CMB: information on intervening structure

- Polarization of the CMB: primordial gravitational waves

- Sunyaev-Zeldovich effect: shadows of galaxy clusters in the CMB

- Combined mass and number of neutrinos.
### 6 Parameter Model of the Universe

#### $\Lambda$CDM

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\Omega_b h^2$</th>
<th>$\Omega_c h^2$</th>
<th>$100\theta_{MC}$</th>
<th>$\tau$</th>
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#### ACDM

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